Friable averages, an overview

Elementary and analytic number theory

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1. Definitions

Let $P^+(n)$ (resp. $P^-(n)$) denote the largest (resp. smallest) prime factor of an integer n with the convention that $P^+(1) = 1$ (resp. $P^-(1) = \infty$).

An integer n is said y-friable if $P^+(n) \leqslant y$.

An integer n is said y-sifted if $P^-(n) > y$.

Canonical representation : $n = ab$ with $P^+(a) \leqslant y$, $P^-(b) > y$.

Friable integers come up in:

- cryptology
- algorithmic theory
- circle method
- sieve theory
- probabilistic number theory (Kubilius' model)
- analytic number theory.

Notation : $S(x,y) := \{n : P^+(n) \leqslant y\}$, $\Psi(x,y) := |S(x,y)|$,
and for $f : \mathbb{N}^* \to \mathbb{C}$,

$$\Psi(x,y;f) := \sum_{n \in S(x,y)} f(n), \quad M(x;f) := \Psi(x,x;f)$$

$$\psi(x,y;f) := \sum_{n \in S(x,y)} \frac{f(n)}{n}.$$

- Daboussi (1984) : $\limsup_{x\to\infty} \frac{|M(x)|}{x} \leqslant \prod_{p \leqslant y} \Big(1 - \frac{1}{p}\Big) \int_1^\infty \frac{|\Psi(x,y;\mu)|}{x^2}\,\mathrm{d}x.$
- Johnsen-Selberg : power-sieve in terms of $\psi(x,y;f)$.

Example: $h(n) := \mathbf{1}_{\square+\square}(n)$, $\sum_{n\leqslant x} h(n) \sim Cx/\sqrt{\log x}$.

An estimate for suitable $\psi(x,y;f)$ by T-Wu (2008) implies

$$\sum_{X<n\leqslant X+N} h(n) \leqslant \{\pi + o(1)\} CN/\sqrt{\log N} \qquad (N \to \infty).$$

2. Available general results

2·1. Via local behaviour

$$\zeta(s,y) := \sum_{P^+(n)\leqslant y} \frac{1}{n^s} = \prod_{p\leqslant y}\Big(1-\frac{1}{p^s}\Big)^{-1}, \quad \varphi_y(s) := \log\zeta(s,y).$$

$\alpha = \alpha(x,y)$ solution of $-\varphi_y'(\alpha) = \log x$.

Hildebrand-T (1986) : *For* $x \geqslant y \geqslant 2$, $u := (\log x)/\log y$, *we have*

$$\Psi(x,y) = \frac{x^\alpha \zeta(\alpha,y)}{\alpha\sqrt{2\pi\varphi_y''(\alpha)}}\Big\{1+O\Big(\frac{1}{u}+\frac{\log y}{y}\Big)\Big\}$$

La Bretèche-T (2005, 2017) various estimates of the shape

$$\Psi(x/d,y) = \{1+O(E)\}\Psi(x,y)/d^\alpha.$$

$$\therefore\ f = g * \mathbf{1} \Rightarrow \Psi(x,y;f) \approx \Psi(x,y)\sum_{d\in S(x,y)} \frac{g(d)}{d^\alpha}.$$

Effective essentially when g small or $\geqslant 0$, hence $f \geqslant 0$.

Examples (La Bretèche-T, 2005): *uniformy for* $x \geqslant y \geqslant 2$, *we have*

(i) $\kappa(n) := \sum_{p|n} \log p$: $\dfrac{\Psi(x,y;\kappa)}{\Psi(x,y)} \sim \dfrac{y \log x}{y + \log x}$,

(ii) $\dfrac{\Psi(x,y,\Omega-\omega)}{\Psi(x,y)} \sim \displaystyle\sum_{p \leqslant y} \frac{1}{p^\alpha(p^\alpha - 1)}$ $\quad [\to \infty \Leftrightarrow y \leqslant (\log x)^{2+o(1)}]$.

Sharpest version of local behaviour (La Bretèche-T, 2017) include first order remainder and coprimality conditions.

2·2. Via mean-value estimates for f(p)

- T-Wu (2003) : $f \geqslant 0$, $\kappa > 0$, and

$(*)$ $\sum_{p \leqslant z} f(p) \log p = \kappa z + O\Big(\dfrac{z}{R(z)}\Big)$, $R(z) \to \infty$, $\displaystyle\int_1^\infty \frac{\mathrm{d}v}{vR(v)\log v} < \infty$.

Then $\Psi(x,y;f) = C_\kappa(f) x \varrho_\kappa(u)(\log y)^{\kappa-1}\{1 + O(E)\}$.

Here $C_\kappa(f) := \prod_p (1 - 1/p)^\kappa \sum_{\nu \geqslant 0} f(p^\nu)/p^\nu$ and $\varrho_\kappa = \varrho^{*k}$.

Error term and validity domain depend on R.

For $R(z) = (\log z)^c$, $c > 1$, then $E = o(1)$ if $\log y \geqslant (\log x \log_2 x)^{2/(3+c)}$.

Hypotheses allow $\mathcal{F}(s) := \displaystyle\sum_{P^+(n)\leqslant y} \frac{f(n)}{n^s} \approx C\zeta(s,y)^\kappa$ only for $s = \alpha_\kappa + i\tau$

where $\alpha_\kappa =$ saddle-point associated to $\zeta(s,y)^\kappa$ bounded τ.

Need to: (a) exploit multiplicativity to link $\Psi(x,y;f)$ to $\psi(x,y;f)$,

(b) get sharp upper bounds,

(c) use an idea of Landau to approximate $\psi(x,y;f)$ by a Perron integral on a bounded segment.

For $b < 3/2$, $s_y := (s-1)\log y$, we have

$$\zeta(s,y) \approx \zeta(s) s_y \widehat{\varrho}(s_y) \qquad \big(\sigma \geqslant 1 - 1/(\log y)^{2b/3},\ |\tau| \leqslant \mathrm{e}^{(\log y)^b}\big).$$

Note: $\widehat{\varrho_\kappa}(s) = \widehat{\varrho}(s)^\kappa$,

$v\varrho_\kappa'(v) - (\kappa-1)\varrho_\kappa(v) + \kappa\varrho_\kappa(v-1) = 0 \qquad (v > 1)$,

$\varrho_\kappa(v) = v^{\kappa-1}/\Gamma(\kappa) \qquad (0 < v \leqslant 1)$.

General theory, Hildebrand-T (1993):

asymptotic, convergent expansion on a “basis” of fundamental solutions.

- T-Wu (2008) :

$$f \geqslant 0,$$

$$(**) \qquad \sum_{p \leqslant z} \frac{f(p)(\log p)}{p} - \kappa \log z \ll 1.$$

$$\psi^*(x, y; f) := \sum_{\substack{n > x \\ P^+(n) \leqslant y}} \frac{f(n)}{n} = \left\{ \mathrm{e}^{-\gamma\kappa} + O(E) \right\} \prod_{p \leqslant y} \sum_{\nu \geqslant 0} \frac{f(p^\nu)}{p^\nu} \int_u^\infty \varrho_\kappa(v)\, \mathrm{d}v$$

where $E = o(1)$ if

$$\frac{\log y}{\sqrt{\log x \log_2 x}} \to \infty, \qquad \sum_p \frac{f(p)^2 (\log p)^2}{p^2} < \infty.$$

Under stronger hypotheses such as estimates on $\sum_{p \leqslant z} f(p) \log p$, domain and accuracy may be significantly improved.

However $(**)$ is adapted to sieve theory.

2·3. *Via assumptions on associated Dirichlet series*

Hanrot–GT-Wu (2006). $\mathcal{F}(s)=\sum_{n\geqslant 1}\frac{f(n)}{n^s}=\prod_{1\leqslant j\leqslant r}\zeta_{\mathbb{K}_j}(s)^{\kappa_j}G(s)$, $0<\beta<\frac{3}{5}$.

$$G(1)\neq 0,\ \ G(s)\ll(1+|\tau|)^{1-\delta}\quad\Big(\sigma>1-\frac{c}{\{\log(3+|\tau|)\}^{(1-\beta)/\beta}}\Big).$$

$$\Psi(x,y;f)=\{1+O(E)\}x\int_{\mathbb{R}}z_\kappa(u-v)\,\mathrm{d}\Big(\frac{M(y^v;f)}{y^v}\Big)\quad\big(\log y>(\log x)^{1-\beta}\big)$$

$\widehat{z_\kappa}(s)=s^{\kappa-1}\widehat{\varrho_\kappa}(s)$,
$vz_\kappa'(v)=-\kappa z_\kappa(v-1)\ (v>1)$, $z_\kappa(v)=1\ (0\leqslant v\leqslant 1)$.

If $\kappa\in\mathbb{N}^*$, the domain may be enlarged to $\log y>(\log_2 x)^{1/\beta}$.

Abstract main term (introduced by de Bruijn): useful for re-summations.

For instance, T-Wu (2008):

$$\sum_{n\leqslant x} f(n)\{\log P^+(n)\}^r = x(\log x)^{\kappa+r-1}\Big\{\sum_{0\leqslant j\leqslant J}\frac{a_j(f)\gamma_j(\kappa,r)}{(\log x)^j}+O\big(\mathcal{R}_J(x)\big)\Big\},$$

$$\mathcal{R}_J(x) := \mathrm{e}^{-(\log y)^{\beta/(1+\beta)}}+\frac{(c_1J+1)^{(J+1)/\beta}}{(\log x)^{J+1}},$$

$\gamma_j(\kappa,r)$ explicit in terms of ϱ_κ,

$$s^\kappa\mathcal{F}(s+1)/(s+1) =: \textstyle\sum_{j\geqslant 0} a_j(f)s^j \qquad (|s|<c).$$

Expansion of main term $x\int_{\mathbb{R}} z_\kappa(u-v)\,\mathrm{d}\big(M(y^v;f)/y^v\big)$: $\forall J\in\mathbb{N}$ and if

$$(\mathcal{D}_J(y))\qquad 0<u<J+2\Rightarrow\langle u\rangle>\varepsilon_{J,y}:=\{(2J+2)\log_2 y\}^{1/\beta}/\log y,$$

$$\Psi(x,y;f)=x\sum_{0\leqslant j\leqslant J}a_j(f)\frac{\varrho_\kappa^{(j)}(u)}{(\log y)^j}+O\bigg(x\varrho_\kappa(u)\Big\{\frac{\log(u+1)}{\log y}\Big\}^{J+1}\bigg)$$

whenever $u\leqslant\begin{cases}(\log y)^{(1-\beta)/\beta} & (\kappa\notin\mathbb{N})\\ \exp\{(\log y)^{1/\beta}\} & (\kappa\in\mathbb{N}).\end{cases}$

Example: $F\in\mathbb{Z}[X]$, $\delta_F(n):=\begin{cases}1 & \text{if } \exists m\in\mathbb{Z}/n\mathbb{Z}: F(m)=0,\\ 0 & \text{otherwise.}\end{cases}$

$$\Psi(x,y;\delta_F)\approx x(\log y)^{\kappa_F-1}\sum_{j\geqslant 0}\frac{a_j(\delta_F)\varrho_{\kappa_F}^{(j)}(u)}{(\log y)^j}$$

if $F(X)=X^3-X-1$, $\kappa_F=5/6$,
if $F(X)=(X^2-2)(X^2-3)(X^2-6)$, $\kappa_F=1$,
if $F=\Phi_\nu$, $\kappa_F=1/\varphi(\nu)$.

For $\kappa \in \mathbb{N}^*$, main term is obtained by writing the Taylor expansion of $z_\kappa(u-v)$ taking the discontinuities into account and estimating the coefficients using the fact that $\int_{\mathbb{R}} v^j \,\mathrm{d}\big(M(y^v;f)/y^v\big)$ converge for every j.

When $\kappa \in \mathbb{R}^+ \smallsetminus \mathbb{N}$, $\vartheta := \langle\kappa\rangle \neq 0$, one has to study

$$\mu_\kappa(v) := \frac{1}{\Gamma(1-\vartheta)} \int_0^{v^+} \frac{M(\mathrm{e}^w;f)}{\mathrm{e}^w(v-w)^\vartheta}\,\mathrm{d}w \qquad (v \in \mathbb{R}).$$

Difficulty: show that, with $\nu := \lfloor\kappa\rfloor$, we have

$$\mu_\kappa(v) = \sum_{0\leqslant j\leqslant\nu} a_{\nu-j}(f)\frac{v^j}{j!} + O\big(\mathrm{e}^{-v^\beta}\big) \qquad (v \geqslant 0).$$

3. New results: oscillating functions

Joint work with la Bretèche (2022).

Dirichlet series $\mathcal{F}(s) = \zeta(s)^{-\kappa}\mathcal{B}(s)$ with $\kappa > 0$,

$\mathcal{B}$ conveniently majorized and has holomorphic continuation to $\sigma > 1 - c/\log(3 + |\tau|)\}^{(1-\beta)/\beta}$ for some $\beta < 3/5$.

$$\Psi(x, y; f) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\mathcal{B}(s, y)x^s}{s\zeta(s, y)^\kappa}\,\mathrm{d}s.$$

Under suitable conditions $\mathcal{B}(s) \approx \mathcal{B}(s, y)$.

Moreover $\zeta(s, y) \approx \zeta(s)s_y\widehat{\varrho}(s_y)$ with $s_y = (s - 1)\log y$.

Let $h_\kappa = z_{-\kappa}$: $vh'_\kappa(v) = \kappa h_\kappa(v - 1)$ $(v > 1)$ and $h_\kappa(v) = 1$ $(0 \leqslant v \leqslant 1)$.

Then $\widehat{h_\kappa}(s) = 1/\{s^{\kappa+1}\widehat{\varrho}(s)^\kappa\}$. Moreover

$$\frac{(s - 1)\mathcal{F}(s)}{s} = \int_0^\infty \mathrm{e}^{-v(s-1)}\,\mathrm{d}\Big(\frac{M(\mathrm{e}^v; f)}{\mathrm{e}^v}\Big) = \int_0^\infty \mathrm{e}^{-vs_y}\,\mathrm{d}\Big(\frac{M(y^v; f)}{y^v}\Big).$$

Hence $s_y \mathcal{B}(s)\widehat{h_\kappa}(s_y)/\{s\zeta(s)^\kappa\}$ is the Laplace transform of

$$J_y(t) := \mathrm{e}^t \int_0^\infty h_\kappa\Big(\frac{t}{\log y} - v\Big)\,\mathrm{d}\Big(\frac{M(y^v; f)}{y^v}\Big).$$

Therefore, it expected that

$$(*) \qquad \begin{aligned} \Psi(x, y; f) &\approx \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\mathcal{B}(s)x^s}{s\zeta(s,y)^\kappa}\,\mathrm{d}s \\ &\approx J_y(\log x) = x\int_0^\infty h_\kappa(u-v)\,\mathrm{d}\mu_{f,y}(v), \end{aligned}$$

with $\mu_{f,y}(v) := M(y^v; f)/y^v$.

This estimate may indeed be derived by the relevant saddle-point method.

However the saddle-point equation fails to have a real solution.

We have to choose the integration abscissa equal to the real part of the two solutions closest to the real line.

Involved error in $(*)$ is $\ll x\mathrm{e}^{-(\log y)^\beta}$.

Approximation of main term.

Case $\kappa \in \mathbb{N}^*$: Taylor expansion of $h_\kappa(u - v)$ taking account of the discontinuities and exploit

$$\frac{a_{j-\kappa-1}(f)}{(\log y)^j} = \frac{(-1)^j}{j!} \int_0^\infty v^j \, \mathrm{d}\mu_{f,y}(v) \qquad (j \geqslant 0),$$

where now $\mathcal{F}(s+1)/\{s^\kappa(s+1)\} =: \sum_{j\geqslant 0} a_j(f)s^j$ $(|s| < c)$.

Under condition $\mathcal{D}_{J+\kappa}(y)$—$\langle u\rangle$ not too close to 0 for $u \leqslant J + \kappa$—we get

$$\Psi(x, y; f) = x \sum_{0\leqslant j\leqslant J} \frac{a_j(f) h_\kappa^{(\kappa+j+1)}(u)}{(\log y)^{\kappa+j+1}} + O\left(\frac{xR_\kappa(u)(\log 2u)^{J+1}}{(\log y)^{\kappa+J+2}}\right)$$

where $R_\kappa(v) = \varrho_\kappa(v)\mathrm{e}^{-\{1+o(1)\}\pi^2 v/2(\log v)^2}$.

Precise description of $h_\kappa^{(j)}$ available from Hildebrand-T (1993).

We have

$$h_\kappa^{(\kappa+j)}(v) = \delta_{0j}\mathrm{e}^{-\gamma\kappa} + O(R_\kappa(v))$$

and an asymptotic series in terms of fundamental solutions may be substituted to the right-hand side.

When $u \leqslant J + \kappa$ and $\langle u \rangle$ is so small that $\mathcal{D}_{J+\nu}(\kappa)$ does not hold, some extra terms should be added to the main term of the asymptotic formula for $\Psi(x, y; f)$: these are $\ll \mathrm{e}^{-(\log(x/y^\ell))^\beta}$ where $\ell := \lfloor u \rfloor$.

When $\kappa \notin \mathbb{N}^*$, the situation becomes much more delicate.

Taylor expansion of $h_\kappa(u - v)$ is inefficient :

(a) the integrals $\int v^j \, \mathrm{d}\mu_{f,y}(v)$ do not converge for large j

(b) asymptotic formula $M(y^v; f) \approx y^v/(v \log y)^{\kappa+1}$ cannot be exploited for small j.

Recall the approximation of the integrand of Perron's formula: $\dfrac{\mathcal{F}(s)}{s s_y^\kappa \widehat{\varrho}(s_y)^\kappa}$.
Let $\nu := \lfloor \kappa \rfloor$, $\vartheta := \langle \kappa \rangle = \kappa - \nu > 0$. We rewrite the above as

$$\frac{s_y \mathcal{F}(s)}{s s_y^\vartheta} \frac{1}{s_y^{\nu+1} \widehat{\varrho}(s_y)^\kappa} =: \widehat{Z_y}(s) \widehat{\varphi_\kappa}(s).$$

$$v\varphi_\kappa'(v) + \vartheta\varphi_\kappa(v) - \kappa\varphi_\kappa(v-1) = 0 \ (v > 1), \ \varphi_\kappa(v) = \frac{v^{-\vartheta}}{\Gamma(1-\vartheta)} \ (0 < v \leqslant 1).$$

$\varphi_\kappa^{(j)}$ has essential discontinuities at $\{0, 1, \dots, j\}$.
General theory still relevant.
All estimates expressed in terms of $\psi_\kappa := \varphi_\kappa^{(\nu)}$.
This function and all its derivatives are over-exponentially decreasing.
The main difficulty: getting suitable upper bounds for (integrals of) $Z_{f,y}$.
For $u = (\log x)/\log y \in \mathcal{D}_{J+\nu}(\kappa, y)$, we get

$$\Psi(x, y; f) = x \sum_{0 \leqslant j \leqslant J} \frac{a_j(f) \psi_\kappa^{(j+1)}(u)}{(\log y)^{\kappa+j+1}} + O\left(\frac{x R_\kappa(u) (\log 2u)^{J+1}}{(\log y)^{\kappa+J+2}} \right).$$

When $u \notin \mathcal{D}_{J+\nu}(y)$ and $\kappa \in \mathbb{N}^*$, some definite quantities $\ll \mathrm{e}^{-(\log(x/y^\ell))^\beta}$ with $\ell := \lfloor u \rfloor$ must be added to the main term.

When $u \notin \mathcal{D}_{J+\nu}(\kappa, y)$, $\ell < u \leqslant \ell+1$, $\ell \leqslant J+\nu+1$, and $\kappa \in \mathbb{R}^+ \smallsetminus \mathbb{N}^*$, formula must be modified by restricting the summation to the (possibly empty) range $0 \leqslant j \leqslant \ell - \nu - 2$ and adding other rapidly decreasing quantities to the main term.

Above results generalise the case $\kappa = 1$ handled in T. (1990) and improve precision.

We thus get, for any given $r < 3/2$, $m := \min(\lfloor u \rfloor, \kappa + 1)$.

$$\Psi(x, y; f) \ll \Psi(x, y; \tau_\kappa)\Big\{\frac{\mathrm{e}^{-cu/(\log 2u)^2}}{(\log y)^{\kappa+m-1}} + \mathrm{e}^{-(\log y)^r}\Big\}.$$

Erdős had conjectured $\Psi(x, y; \mu)/\Psi(x, y) \to 0$, proved by Alladi (1982) and Hildebrand (1984, 1987).

4. Applications

4·1. Weighted averages

For $0 < \beta < 3/5$, $J \in \mathbb{N}$, f as above, and $\mathrm{e}^{(\log x)^\beta} \leqslant y \leqslant x$, $u \in \mathfrak{D}_{J+\nu}(y)$,

$$m(x, y; f) := \sum_{n \in S(x,y)} \frac{f(n)}{n} = \sum_{0 \leqslant j \leqslant J} \frac{a_j^*(f)\psi_\kappa^{(j)}(u)}{(\log y)^{\kappa+j}} + O\bigg(\frac{R_\kappa(u)(\log 2u)^J}{(\log y)^{J+\kappa+1}}\bigg),$$

with $a_j^*(f) := a_j(f) + a_{j-1}(f)$ $(j \geqslant 0)$.

4·2. Truncated multiplicative functions

Define $f_y(p^\nu) = f(p^\nu)$ $(p \leqslant y)$, $:= 1$ $(p > y)$.

With $g := f * \mu$ (associated to $\zeta(s)^{-\kappa-1}$), we have

$$M(x; f_y) = \sum_{n \in S(x/z,y)} g(n) \left\lfloor \frac{x}{n} \right\rfloor + \sum_{d \leqslant z} M\Big(\frac{x}{d}, y; g\Big) - M\Big(\frac{x}{z}, y; g\Big) \lfloor z \rfloor.$$

Let $\mathcal{D}_J(b,y) := \Big\{u \geqslant 1 : \min_{1\leqslant j<\min(u,J+1)}(u-j) > 1/(\log y)^b\Big\}$.
Using the results previously described, we obtain that for

$$0<\beta<1/2,\ \ \log y > (\log x)^{1-\beta},\ \ b := \frac{1-2\beta}{1-\beta},\ \ J\geqslant 0,\ \ u\in\mathcal{D}_{J+\nu+1}(b,y),$$

we have

$$M(x;f_y) = x\sum_{0\leqslant j\leqslant J}\frac{a_j(f)\psi^{(j)}_{\kappa+1}(u)}{(\log y)^{\kappa+j+1}} + O\bigg(\frac{xR_\kappa(u)(\log 2u)^{J+1}}{(\log y)^{J+\kappa+2}}\bigg).$$

In special case $f(n) := (-k)^{\omega(n)}$, $k\in\mathbb{N}^*$, this improves on Alladi-Goswami (2022) who had β arbitrarily small and only the dominant term in the expansion.

For this function we have $a_0(f)=0$ whenever $k=p+1$ for some prime p and so the first term of the expansion vanishes in this case.