# DENSITY THEOREMS FOR THE RIEMANN ZETA FUNCTION IN THE NEIGHBOURHOOD OF THE LINE $\mathrm{Re}\ s = 1$

JÁNOS PINTZ

RÉNYI INSTITUTE, BUDAPEST

**1.**

$$N(\sigma, T) := \sum_{\substack{\varrho=\beta+i\gamma;\, \zeta(\varrho)=0 \\ \beta\ge\sigma,\, 0<\gamma<T}} 1, \quad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

First density theorem

Carlson (1920): $N(1-\eta, T) \ll_\varepsilon T^{4\eta(1-\eta)+\varepsilon}$ for $\eta < 1/2$.

Significance: $N(1-\eta, T) \ll_\varepsilon T^{A\eta+\varepsilon} \Rightarrow p_{n+1} - p_n \ll n^{1-1/A+\varepsilon}$.

Best conjecture: Density Hypothesis $\Longleftrightarrow A = 2$.

Best result Huxley (1972): $A = 12/5$.

First proof of the Density Hypothesis for some interval $c_0 \geq \eta \geq 0$:

**Theorem A** (Halász–Turán, 1969, improved form in Turán's book).

$$(1.1) \qquad N(1-\eta, T) \ll_\varepsilon T^{A\eta^{3/2}} \log^c T, \qquad A = 12.000.$$

Main ideas of the proof:

(1.2)

(i) $\zeta(1-\eta, T) \ll T^{B\eta^{3/2}} \log^c T$;

(ii) Turán's power-sum method;

(iii) an idea of Halász.

Different proofs (by large sieve + Halász' idea, avoiding Turán's method) were given by Montgomery and Bombieri.

Montgomery: $A = \frac{40}{3} B$ can be chosen in (1)–(2).

Richert (1967): $B = 100 \Longrightarrow A = 4000/3 + \varepsilon$.

Ford (2000): $B = 4.45 \Longrightarrow A = 58.05$

Heath-Brown (2017): $A = 6.42$ for $\eta \le 1/10$, $A = 5.03$ for $\eta < \eta_0$ if $\log^c T$ is substituted by $T^\varepsilon$ in (1.1).

The proof is based on a new form of Vinogradov's mean value theorem proved by Wooley and Bourgain, Demeter and Guth (2016).

Various explicit zero density theorems for $\sigma$ near to 1 (i.e. for small values of $\eta = 1 - \sigma$) were proved by Ivič, Jutila, Heath-Brown, Huxley, Bourgain, of the form

$$N(1-\eta, T) \ll T^{A(\eta)\eta} \log^c T \quad \text{or} \tag{1.3}$$

$$N(1-\eta, T) \ll_\varepsilon T^{A'(\eta)\eta+\varepsilon}, \tag{1.4}$$

for example,

(1.5) Ivič (1979): $A(\eta) = 35/36$ if $\eta < 3/155$,

(1.6) Bourgain (1995): $A'(\eta) = 4\big/(5-30\eta)$ for $\eta < 1/16$,

(1.7)
Bourgain (1995): $A'(\eta) = 2\big/(2-7\eta)$ for $1/16 \le \eta \le 2/19$.

## 2.

We will sketch the proof of a theorem which

(a) improves the result of Bourgain for $\eta < 1/20$;
(b) gives an explicit zero-density estimate of type (1.3) for every $\eta < 1/12$ (with different forms of $A(\eta)$ in different intervals);
(c) improves the Halász–Turán type density theorems of Heath-Brown.

Notation: For $k \geq 4$, $\ell \geq 3$,

(2.1)

$$I(k,\ell) := \left[\frac{1}{2\ell(\ell+1)}, \frac{1}{2\ell(\ell-1)}\right) \bigcap \left[\frac{1}{k(k+1)}, \frac{1}{k(k-1)}\right).$$

**Remark 1.** In Theorem 7 and Corollaries 1–2 the constant in the $\ll$ sign may depend on $\eta$.

**Theorem 1.** *The density estimate* (1.3) *holds for* $\eta \in I(k, \ell)$ *with*

(2.2)

$$A'(\eta) = \max\left\{\frac{3}{\ell(1-2(\ell-1)\eta)}, \frac{4}{k(1-(k-1)\eta)}\right\} \quad \text{if } \eta < 1/12.$$

**Remark 2.** Since the first term is larger for most values of $\eta$:

## Theorem 1'

*If* $\eta < 1/12$, $\eta \notin [1/42, 1/40]$, $\eta \in [1/2\ell(\ell+1), 1/2\ell(\ell-1)]$ *then*

$$A'(\eta) = \frac{3}{\ell(1-2(\ell-1)\eta)}. \tag{2.3}$$

**Corollary 1.** *The density estimate* (1.3) *holds for* $\eta \in I(k,\ell)$ *with*

$$(2.4) \qquad A'(\eta) = \max\left\{\frac{3}{\ell-1}, \frac{4}{k-1}\right\} \qquad \textit{if} \quad \eta < 1/12.$$

**Corollary 2.** *If* $\varepsilon > 0$ *then* $N(1-\eta, T) \ll_\varepsilon T^{(3\sqrt{2}+\varepsilon)\eta^{3/2}+\varepsilon}$ *if* $\eta < \eta_0(\varepsilon)$ *is sufficiently small.*

## 3. Main ingredients of the proof

(a) Exponential sum estimates of Heath-Brown [Hea] which appeared in Proc. Steklov Institute, 2017. These are based on an (essentially) optimal form of Vinogradov's mean-value theorem proved by Wooley and Bourgain, Demeter and Guth (2016).

**Theorem 1 of [Hea].** *Let $k \geq 3$ be an integer, $f(x) : [0, N] \to R$ having continuous derivation of order up to $k$ on $(0, N)$ with*

$$(3.1) \qquad 0 < \lambda_k \leq f^{(k)}(x) \leq A_k \lambda_k, \quad x \in (0, N).$$

*Then (with $e(x) = e^{2\pi i x}$)*

(3.2)

$$\frac{1}{N} \sum_{n \leq N} e(f(n))$$

$$\ll_{A,k,\varepsilon} N^{\varepsilon} \left( \lambda_k^{1/k(k-1)} + N^{-1/k(k-1)} + N^{-2/k(k-1)} \lambda_k^{-2/k^2(k-1)} \right).$$

**Theorem 5 of [Hea].** *For $t \geq 1$ and $1/2 \leq \sigma \leq 1$ we have with $\kappa = 8\sqrt{15}/63 = 0.4918\ldots$*

$$\zeta(\sigma + it) \ll_\varepsilon t^{\kappa(1-\sigma)^{3/2}+\varepsilon}. \tag{3.3}$$

(b) A simple but ingenious idea of Halász.

(c) We will also use a method used originally to show

$$|M(X)| = \left|\sum_{n \leq X} \mu(n)\right| \geq c_0\sqrt{X} \quad \text{for} \quad X > X_0 \quad \text{effectively.}$$

**Corollary I of Theorem 1 of [Hea].** *If* $I(N) \subset [N, 2N)$, $t \in [3, T]$, $r \geq 2$

(3.4)
$$|t|^{2/r} \gg N \geq N_r(\xi, T) := T^{1/r(1-(r-1)\xi-6r\varepsilon)}, \ \ \xi \leq 1/r(r-1)-6\varepsilon,$$

*then*

$$\sum_{n \in I(N)} n^{-(1-\xi)+it} \ll_{\varepsilon,r} T^{-\varepsilon}. \tag{3.5}$$

**Theorem II.2 of [HLM].** *If $s = \sigma + it$, $\sigma \geq 1 - \xi$, $\xi > 0$, $-\xi < \theta \leq 1$ then*

$$(3.6) \qquad \sum_{n=1}^{\infty} \left(e^{-n/2N} - e^{-n/N}\right) n^{-s} \ll N^{\theta - 1 + \xi} \max_{\sigma' \geq \theta, 1 < |t'| \leq 2|t|} |\zeta(\sigma' + it')| + N^{\xi} e^{-|t|}.$$

This is Theorem II.2 in the Appendix of Montgomery's book.

## 4. Sketch of proof of Theorem 1

Let us consider the zeros $\varrho_j = \beta_j + i\gamma_j$ with $\beta_j = 1 - \eta_j \geq \sigma := 1 - \eta$, $2 \log T < |\gamma_j| \leq T$ $(j = 1, 2, \ldots, K)$, and let us choose with a sufficiently small $\varepsilon > 0$ $\big(\varepsilon < \varepsilon_0(k, \ell, \eta)\big)$, $T > T_0(\varepsilon)$

(4.1)

$$X := T^{\varepsilon}, \quad Y := T^{2/k}, \quad Y_1 := Ye^3, \quad \lambda = \log Y, \ \mathcal{L} = \log T,$$

$$M_X(s) := \sum_{n \leq X} \frac{\mu(n)}{n^s},$$

$$a_n := \sum_{d|n, d \leq X} \mu(d), \ N_k^*(T) = XN_k(\xi, T).$$

Let

(4.2)

$$I_j := \frac{1}{2\pi i} \int_{(3)} \sum_{n=1}^{\infty} \frac{a_n}{n^{s+\varrho_j}} \frac{e^{s^2/\lambda+\lambda s}}{s} ds$$

$$= \frac{1}{2\pi i} \int_{(3)} M_X(s+\varrho_j)\zeta(s+\varrho_j)\frac{e^{s^2/\lambda+\lambda s}}{s} ds$$

$$= \frac{1}{2\pi i} \int_{(-\eta)} M_X(s+\varrho_j)\frac{\zeta(s+\varrho_j)}{s} e^{s^2/\lambda+\lambda s} ds$$

$$= \frac{1}{2\pi} \int_{-2\lambda}^{2\lambda} M_X(1-\eta-\eta_j+i(\gamma_j+t)) \cdot$$

$$\cdot \frac{\zeta(1-\eta-\eta_j+i(\gamma_j+t))}{-\eta+it} e^{(-\eta+it)^2/\lambda} Y^{-\eta+it} dt + O\left(\frac{1}{Y^3}\right).$$

Shifting the part of the series $\sum_n a_n n^{-(s+\varrho_j)}$ with $n \geq Y_1$ (i.e. $\lambda - \log n \leq -3$) from Re $s = 3$ to $\mathrm{Re}\ s = \lambda$ and the finite part $n \in (X, Y_1)$ to $\mathrm{Re}\ s = 1/\lambda$, the term $n = 1$ to $\mathrm{Re}\ s = -4$ we obtain

(4.3)

$$I_j = 1 + \sum_{X<n<Y_1} a_n n^{-\varrho_j} \frac{1}{2\pi i} \int_{1/\lambda-2i\lambda}^{1/\lambda+2i\lambda} \frac{e^{s^2/\lambda+(\lambda-\log n)s}}{s} ds + O\left(\frac{1}{Y^3}\right).$$

The RHS of (4.2), the estimation (3.3) for the zeta-function imply

$$(4.4) \qquad I_j \ll T^{\varepsilon+\kappa(2\eta)^{3/2}} \cdot Y^{-\eta} \ll T^{10^{-4}\eta/k},$$

since $\sqrt{\eta} \leq 2/\sqrt{5}k$ by $\eta \leq 1/k(k-1)$.

The formulae (4.1)–(4.3) imply that with a suitable choice of $\gamma_j^* \in (\gamma_j - 2\lambda, \gamma_j + 2\lambda)$ we have with $\varrho_j^* = 1 - (\eta_j - 1/\lambda) + i\gamma_j^*$

$$(4.5) \qquad I_j^* := \Big| \sum_{X<n<Y_1} a_n n^{-\varrho_j^*} \Big| > 1/2\lambda \quad (j = 1, 2, \ldots K).$$

We will select a subset of $\{\varrho_j^*\}$ with $K' \gg K/\lambda\mathcal{L}$ elements and $|\gamma_\nu^* - \gamma_\kappa^*| > 2\lambda$. Furthermore, we divide $a_n$ for all $n \in (X, Y_1)$ into two parts (for $N_k(T) = N_k(\eta, T)$ see Corollary I):

$$(4.6)\quad a_n' = \sum_{\substack{d,m \\ dm=n \\ m>N_k(T) \\ d\le X}} \mu(d), \quad a_n'' = \sum_{\substack{d,m \\ dm=n \\ m\le N_k(T) \\ d\le X}} \mu(d), \quad a_n = a_n' + a_n''.$$

By (3.5) we obtain

$$(4.7)\qquad \sum_{X<n<Y_1} a'_n n^{-\varrho_j^*} = \sum_{d\le X}\mu(d)d^{-\varrho_j^*}\sum_{\substack{X/d<m<Y_1/d\\ m>N_k(T)}} m^{-\varrho_j^*} \ll_{k,\varepsilon} XT^{-2\varepsilon} = T^{-\varepsilon}.$$

By (4.5) and (4.7) we have a $U \in (X, N_k(T)X)$ such that for some $I(U) \subset [U, 2U]$ and with $a''_n$ in (4.6)

$$(4.8)\qquad \frac{K'}{3\lambda^2} \le \sum_{\nu=1}^{K'}\Big|\sum_{n\in I(U)} a''_n n^{-\varrho_\nu^*}\Big| =: \sum_{\nu=1}^{K'}\Big|\sum\nolimits_\nu\Big|.$$

Let $U = T^u$. An easy calculation shows
(4.9)
$U \le XN_k(T) = N_k(T)T^\varepsilon \Rightarrow u \le u(\ell,\eta) := 1\big/\ell(1-2\eta(\ell-1)-6\ell\varepsilon)$.

If $u > u(\ell,\eta)\big/2$ we raise $\Sigma_\nu$ to the $2^{\text{nd}}$ power, $U^2 \le N_k^2(T)T^{2\varepsilon}$. If $u \le u(\ell,\eta)\big/2$ we raise $\Sigma_\nu$ to the $h^{\text{th}}$ power such that

$$hu \in \big[u(\ell,\eta),(3/2)u(\ell,\eta)\big] =: J_\ell(\eta). \qquad (4.10)$$

$\Sigma_\nu^h$ runs from $A_h$ to $B_h$ where $U^h \le A_h \le B_h \le (2U)^h$.

We will denote the resulting coefficients after taking $h^{\text{th}}$ power of $\sum_\nu$ (where now $h \in [2, 20/\varepsilon))$ by $a_n^{(h)} = b_n$ and will find $\alpha_\nu$ $(\nu = 1, 2, \ldots, K')$ with $|\alpha_\nu| = 1$ satisfying

$$\left|\sum\nolimits_\nu^h\right| = \Big|\sum_{A_h \le n \le B_h} b_n n^{-\varrho_\nu^*}\Big| = \alpha_\nu \sum_{A_h \le n \le B_h} b_n n^{-\varrho_\nu^*}. \tag{4.11}$$

Using Halász' idea, after taking the $h^{\text{th}}$ powers of $\sum_\nu$ we square both sides and by the Cauchy–Schwarz inequality we get

(4.12)

$$\frac{(K')^2}{\mathcal{L}^{2h}} \ll_\varepsilon \left( \sum_{A_h \le n < B_h} \frac{|b_n|^2}{n} \right) \left( \sum_{A_h \le n < B_h} \sum_{\nu,\kappa=1}^{K'} \alpha_\nu \overline{\alpha}_\kappa \frac{1}{n^{1-\eta_\nu-\eta_\kappa+i(\gamma_\nu^*-\gamma_\kappa^*)}} \right)$$
$$\ll \mathcal{L}^{c(\varepsilon)} \left\{ K'(K'-1)T^{-\varepsilon} + K' B_h^{2\eta} \right\}$$

by $2\ell(\ell-1)\eta < 1$. The crucial estimate $T^{-\varepsilon}$ for the exponential sum in the second factor follows from (3.5) if for the corresponding pair $(\nu, k)$ we have $B_h \le \left|\gamma_\nu^* - \gamma_k^*\right|^{2/\ell}$. Otherwise the same estimate $T^{-\varepsilon}$ follows from (3.3) and a variant of Theorem II.2 of the Appendix of Montgomery's book (cf. (3.6)).

Hence, using $B_2 \leq 4\big(N_k^*(T)\big)^2$ and $hu \in J_\ell(\eta)$ we obtain

(4.13)
$$K' \ll \mathcal{L}^{2c(\varepsilon)} B_h^{2\eta} \ll T^{\eta(\max(3/\ell(1-a\eta(\ell-1)-2\ell\varepsilon),4/k(1-(k-1)\eta)-2k\varepsilon))}.$$

Consequently,

(4.14)
$$K \ll T^{A'(\eta)\eta+\varepsilon}.$$

Thank you for your attention.