Classification of L-functions of degree 2 and conductor 1

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• **Framework:** *Extended S-class* $\mathcal{S}^\sharp$:
- Dirichlet series for $\sigma > 1$,
- general funct. eq. with multiple Γ-factors;
(more details later on)

S-class $\mathcal{S}$: $F \in \mathcal{S}^\sharp$ with
- general Euler product,
- Ramanujan conj.

• **What do S-classes contain ?** Difficult problem; *general expectation*:
- $d \notin \mathbb{N} \to$ no functions of **degree** d in $\mathcal{S}^\sharp$,
- $d \in \mathbb{N} \to \{F \in \mathcal{S}$ of degree $d\}$
$= \{$automorphic L-functs. of degree $d\}$,
- $F \in \mathcal{S}^\sharp$ of degree $d \in \mathbb{N} \to$???

$\mathcal{S}$ and $\mathcal{S}^\sharp$ known for degree $d < 2$ (Conrey-Ghosh 1993, Kac.-Per. 1999-2011), confirming expectation and describing $F \in \mathcal{S}^\sharp$ with $d = 0$ (*suitable D-polyn.*) and $d = 1$ (*suitable lin. comb. of* $L(s,\chi)$*'s*).

First open case $d = 2$; here expect:
- $F \in \mathcal{S} \to L$-funct. of Hecke or Maass eigenforms of any level;
- $F \in \mathcal{S}^\sharp \to$??? (Hecke's "triangle forms"?).

• **Degree** $d = 2$ **and conductor** $q = 1$**.** General case $d = 2$ apparently very difficult. Next important invariant after degree is **conductor** q (e.g. $q =$ *level* for modular L); if $q = 1$ nice phenomenon happens allowing complete description.

Classification by *new invariant* χ_F (**eigenweight**) and requires *normalization* of F to fit "modular" framework. However:
- every F with $d = 2$ and $q = 1$ can be normalized in a simple way (vertical shift + divide by first coeff. $\neq 0$);
- χ_F easy to compute from data of F.
For example:

$$\chi_F = 0 \Longrightarrow F = \zeta^2, \qquad \chi_F = \frac{121}{2} \Longrightarrow F = L_\Delta$$

• **Theorem.** *Let* $F \in \mathcal{S}^\sharp$ *with* $d = 2$, $q = 1$ *and normalized. Then* $\chi_F \in \mathbb{R}$ *and*
$\chi_F > 0 \Rightarrow F = L_f$*, with* f *Hecke cusp form of level 1 and even integral weight* $k = 1 + \sqrt{2\chi_F}$*;*
$\chi_F = 0 \Rightarrow F = \zeta^2$*;*
$\chi_F < 0 \Rightarrow F = L_u$*, with* u *Maass form of level 1, weight 0 and eigenvalue* $1/4 + \kappa^2 = (1 - 2\chi_F)/4$*.*

When $\chi_F < 0$, parity of u is $\varepsilon = \frac{1-\omega_F}{2}$, $\omega_F =$ **root number** of F.

Theorem confirms expectation (linear independence in $\mathcal{S}$).

• Some definitions and properties.
Class $\mathcal{S}^\sharp$ and invariants:

$$\gamma(s)F(s) = \omega\overline{\gamma(1-\overline{s})}\,\overline{F(1-\overline{s})}, \quad |\omega| = 1,$$

$$\gamma(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j);$$

γ-factor $\gamma(s)$ has $Q > 0, \lambda_j > 0, \Re(\mu_j) \geq 0$.

$$d = 2\sum_{j=1}^{r} \lambda_j, \quad q = (2\pi)^d Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j},$$

$$\omega_F = \omega \prod_{j=1}^{r} \lambda_j^{-2i\Im\mu_j}, \quad H(n) = 2\sum_{j=1}^{r} \frac{B_n(\mu_j)}{\lambda_j^{n-1}}$$

H-**invariants** ($H(0) = d$), eigenweight:

$$\chi_F = H(1) + H(2) + 2/3 \text{ (easy to comp.).} \quad (1)$$

For normalized F with $d = 2$ and $q = 1$:
- **invariant form of funct. eq.** (Γ-reflection formula + real D-coefficients):

$$F(s) = S_F(s)h_F(s)F(1-s) \quad (2)$$

$$S_F(s) := 2^r \prod_{j=1}^{r} \sin(\lambda_j s + \mu_j) = \sum_{j=0}^{N} a_j e^{i\pi\omega_j s}$$

($a_j \neq 0$, $-1 = \omega_0 < ... < \omega_N = 1$, $\omega_j = -\omega_{N-j}$)

$$h_F(s) \approx \frac{\omega_F}{\sqrt{2\pi}}(4\pi)^{2s-1}\sum_{\ell=0}^{\infty} d_\ell \Gamma(2(s_\ell - s)) \qquad (3)$$

$\approx$ asympt. exp., where d_ℓ **structural invariants** (complicated recursive def., $d_0 = 1$) and $s_\ell = 3/4 - \ell/2$; $h_F(s)$ and $S_F(s)$ are *invariants*;

- **standard twist**:

$$F(s,\alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-\alpha\sqrt{n}), \;\; \alpha > 0, e(x) = e^{2\pi i x},$$

Spec$(F) = \{2\sqrt{m} : m \in \mathbb{N}$ with $a(m) \neq 0\}$;
$\alpha \in$Spec$(F) \Rightarrow F(s,\alpha)$ has at most simple poles at $s = s_\ell$ ($\ell \geq 0$) with residue

$$\rho_\ell(\alpha) = d_\ell \frac{e^{i\pi/4}\overline{a(\alpha^2/4)}}{(-2\pi i)^\ell \alpha^{\ell+1/2}} \quad (\rho_0(\alpha) \neq 0). \qquad (4)$$

REMARK. Conj. by Kac.-Per. (2002):
funct. eq. of $F \in \mathcal{S}^\sharp$ of degree d is determined by q, ω_F and $H(n)$ with $n \leq d$.
Theorem confirms this when $d = 2$ and $q = 1$, in view of definition of χ_F in (1)

• **Basic ideas of proof.** Four steps.
1. Transformation formula and invariants.
Structural invariants d_ℓ appear in:
- funct. eq. of F, see (2) and (3)
- residues of standard twist, see (4).

But: special form of transformation formula for standard twists when F normalized with $d = 2$ and $q = 1$ $\implies$ *every d_ℓ ($\ell \geq 2$) determined by d_1 by algorithm independent of F.*

So $d_1 \to h_F(s)/\omega_F$, and computation shows that

$$\chi_F = d_1 + \frac{1}{8} \in \mathbb{R}; \qquad (5)$$

hence:

value of χ_F determines $h_F(s)/\omega_F$.

2. Virtual γ-factors.
Virtual γ-factors of Hecke and Maass type:

$(2\pi)^{-s}\Gamma(s+\mu) \quad \mu > 0$

$\pi^{-s}\Gamma\Big(\frac{s+\varepsilon+i\kappa}{2}\Big)\Gamma\Big(\frac{s+\varepsilon-i\kappa}{2}\Big) \quad \varepsilon \in \{0,1\},\ \kappa \geq 0.$

Although such $\gamma(s)$ not always associated with L-function, their struct. invariants d_ℓ have *same formal properties* as in $\mathcal{S}^\sharp$ ($d_1 \to d_\ell$). Moreover

$$\chi_\gamma = \begin{cases} 2\mu^2 \\ -2\kappa^2, \end{cases}$$

hence $\{\chi_\gamma : \gamma(s)$ virtual γ-factor$\} = \mathbb{R}$. Thus to $F \in \mathcal{S}^\sharp$ associate virtual γ-factor $\gamma(s)$ such that

$$\chi_\gamma = \chi_F.$$

Therefore $h_F(s) = \omega_F h_\gamma(s)$, hence by (2) funct. eq. of F becomes

$$\gamma(s)F(s) = \omega_F R(s)\gamma(1-s)F(1-s), \qquad (6)$$

$$R(s) = \frac{S_F(s)}{S_\gamma(s)} \quad \text{(satisfying } R(s)R(1-s) = 1\text{)}.$$

REMARK. If $R(s) =$ const., then Theorem follows from classical converse theorems of Hecke and Maass. Moreover

$$R(s) \neq \text{const.} \Longrightarrow N \geq 3 \text{ and } \omega_{N-1} > 0. \qquad (7)$$

3. Period functions.

Proving that $R(s) =$ const. quite involved. Based on study of associated "modular form"

$$f(z) = \sum_{n=1}^{\infty} a(n)n^\lambda e(nz) \quad z \in \mathbb{H},\ \lambda = \begin{cases} \mu \text{ (H-case)} \\ i\kappa \text{ (M-case)} \end{cases}$$

and **period function** (Lewis-Zagier 2001)

$$\psi(z) = f(z) - z^{-2\lambda-1} f(-1/z).$$

By study of $f(z)$ get:

$i)$ $\psi(z)$ *holomorphic for* $-\rho\pi < \arg(z) < \pi$ $(\rho > 0)$,

$$ii)\ \psi(z) = Q(z) + \textit{ holom. for } |\arg(z)| < \pi \quad (8)$$

with certain integral $Q(z)$,

$$iii)\ \psi(z) = \psi(z+1) + (z+1)^{-2\mu-1}\psi\Big(\frac{z}{z+1}\Big)$$

(three-term funct. eq.).
Hence by i)–iii) get

$$\psi(z) \textit{ and } Q(z) \textit{ holom. for } -\pi < \arg(z) < \pi. \quad (9)$$

REMARK. Maass case analogous to Lewis-Zagier; Hecke case more delicate, involving use of special functions.

4. Conclusion of proof. By contradiction, assuming $R(s) \neq$ const. Then:
- using (7), integral $Q(z)$ in (8) transformed to (roughly)

$$f(z) + \text{ holom. for } -\rho'\pi < \arg(z) < \pi \ (\rho' > 0),$$

- hence (9) $\Longrightarrow$

$$f(z) \text{ holomorphic for } -\rho'\pi < \arg(z) < \pi.$$

But $f(z)$ is 1-periodic, so $f(z)$ *entire* and hence $f \equiv 0$, contradiction.

• Some details for step 1.

General transf. formula for nonlinear twists:

$$F(s;f) := \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-f(n,\boldsymbol{\alpha}))$$

$$= \sum_{j=0}^{J} W_j(s,\boldsymbol{\alpha}) \overline{F}(s^* + \eta_j; f^*) + \text{ holom.},$$

f - nonlinear twist, $f(n,\boldsymbol{\alpha}) = \sum_{j=0}^{M} \alpha_j n^{\kappa_j}$

f^* - its *dual* twist,

W_j - holom., $W_0(s) \neq 0$,

s^*- certain linear funct. of s,

η_j - shifts with $0 = \eta_0 < \eta_1 < ... < \eta_J$.

In general *no analytic info on* $F(s;f)$ and $F(s;f^*)$, but if $F \in \mathcal{S}^\sharp$ normalized with $d = 2$, $q = 1$, $\alpha \in$ Spec(F) and choosing

$$f(n,\alpha) = n + \alpha\sqrt{n} \quad \text{(thus } F(s;f) = F(s,\alpha)\text{)},$$

then transf. formula simplifies to

$$F(s,\alpha) = \sum_{j=0}^{J} W_j(s,\alpha) F(s + j/2, \alpha) + \text{holom.}, \tag{10}$$

W_j explicit (complicated) polyn. involving structural invariants d_ℓ, $W_0 \equiv 1$.

From (10) obtain

$$\sum_{j=1}^{J} W_j(s,\alpha)F(s+j/2,\alpha) = \text{holom.},$$

so, computing residues, by (4) get: $\forall N \geq 2\ \exists$

$$Q_N(X_0,...,X_N) = \sum_{\ell,h\geq 0,\ell+h\leq N} \alpha_{\ell,h} X_\ell X_h$$

with $\alpha_{\ell,h} \in \mathbb{R}$, $\alpha_{0,N} + \alpha_{N,0} = 1$ *s.t. for every* F

$$Q_N(d_0,...,d_N) = 0. \tag{11}$$

From shape of (11) get: d_1 *determines all* d_ℓ *with* $\ell \geq 2$, hence by (5) so does χ_F.

REMARK. We believe similar phenomenon holds in general, i.e.: *invariants* d_ℓ *should lie on algebraic varieties largely independent of* F. This could explain why Γ-factors of L-functions have special shape.

• Some details for step 2.

$\gamma(s)$ virtual γ-factor. Computation from definition of d_ℓ shows:

$$\forall\ell\ \exists P_\ell, Q_\ell \in \mathbb{R}[x] \text{ s.t. } d_\ell = \begin{cases} P_\ell(\mu) & (\textit{H-case}) \\ Q_\ell(\kappa) & (\textit{M-case}). \end{cases}$$

If $\gamma(s)$ associated to L-function, then (11) holds for such d_ℓ. But this is polyn. eq. in μ or κ, and $\exists\infty$-many μ (weights) and κ (eigenvalues). So (11) holds identically in μ or κ, hence d_1 determines all d_ℓ for virtual γ-factors as well.

Hence χ_γ determines $h_\gamma(s)$ and step 2 follows.

• Some details for step 3.
Let $z = x + iy$, $y > 0$. Start with Mellin's transf. and use funct. eq. (6) in step 2 $\rightarrow$ express $f(iy)$ as (roughly)

$$\int_{(c)} \Gamma(s) \frac{\gamma(1-s+\lambda)}{\gamma(s-\lambda)} R(1-s) F(1-s+\lambda)(2\pi y)^{-s} ds,$$

$\gamma(s)$ - virtual γ-factor of F, $R(s) = S_F(s)/S_\gamma(s)$.

Hecke case (Maass case simpler thanks to shape of virtual $\gamma(s)$). Using

$$S_F(s) = -2\omega_F \cos(\pi s) + \sum_{j=1}^{N-1} a_j e^{i\pi\omega_j s},$$

after expansion of F + manipulations get (roughly)

$$f(iy) = y^{-\mu-1} \sum_{n=1}^{\infty} a(n) J\Big(\frac{2\pi n}{y}\Big) + Q(iy) \qquad (12)$$

where

$$J(w) = \frac{1}{2\pi i} \int_{(1+\delta)} \frac{1}{\cos(\pi s)\Gamma(1-s-\mu)} w^{-s} dw,$$

and

$$Q(z) \ \textit{holom.}\ -\pi(1-\omega_{N-1}) < \arg(z) < \pi.$$

Moreover

$$J(w) = -\frac{w^{-1/2}}{\pi} E_\beta(w) \qquad (13)$$

with $E_\beta(w)$ Mittag-Leffler function with $\beta = 1/2 - \mu$, satisfying

$$E_\beta(w) = \kappa_0 e^{-w} w^{1-\beta} + I_\beta(w), \qquad (14)$$

$I_\beta(w)$ *holom. on* $\mathbb{C}$ *with suitable cut*.

But first term of (14) with $w = 2\pi n/y$ rebuilds $f(i/y)$, so from (12)-(14) arrive to

$$\begin{aligned} f(z) &= z^{-2\mu-1} f(-1/z) + Q(z) + \text{ holom.} \\ &= z^{-2\mu-1} f(-1/z) + \psi(z), \end{aligned} \qquad (15)$$

$\psi(z)$ *holom.* $-\pi(1 - \omega_{N-1}) < \arg(z) < \pi$.

Moreover, from (15) + 1-periodicity of $f(z)$ get three-term funct. eq. and then continuation of $\psi(z)$ and $Q(z)$ to $|\arg(z)| < \pi$.

REMARK. Actually, continuation of $Q(z)$ to any sector larger than $-\pi(1-\omega_{N-1}) < \arg(z) < \pi$ enough to conclude proof (will see in next step).

• Some details for step 4.

By contradiction, assume $R(s) \neq$ const., so

$$N \geq 3 \textit{ and } \omega_{N-1} > 0 \textit{ by } (7).$$

Recall $Q(z)$ sum of integrals over $j = 1, ..., N-1$ and accordingly write

$$Q(z) = \sum_{j=1}^{N-2} Q_j(z) + Q_{N-1}(z).$$

Then (roughly):

$$\textit{sum holom. } -\pi(1-\omega_{N-2}) < \arg(z) < \pi;$$

$$Q_{N-1}(z) \textit{ rebuilds to } f(e^{i\pi(1-\omega_{N-1})}z).$$

But $Q(z)$ holom. $|\arg(z)| < \pi$ and $\omega_{N-2} < \omega_{N-1}$, hence

$$\text{f(z) } \textit{holom. } -\delta\pi < \arg(z) < \pi \textit{ with } \delta > 0,$$

therefore $f(z)$ entire by 1-periodicity, thus $f \equiv 0$, contradiction.