# On autocorrelations of some aperiodic multiplicative functions

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Based on a joint work with Alex Gomilko and Thierry de la Rue

- (S) $\lim_{N\to\infty} \frac{1}{N}\sum_{n\leqslant N} f(T^n x)\boldsymbol{\mu}(n) = 0$ for all dynamical systems $(X, T)$ with $h(T) = 0$, $f \in C(X)$ and $x \in X$ (Sarnak, 2010).
- What about other multiplicative functions? Say $\boldsymbol{u} : \mathbb{N} \to \mathbb{D}$.
- $\lim_{N\to\infty} \frac{1}{N}\sum_{n\leqslant N} f(T^n x)\boldsymbol{u}(n) = 0$ for all dynamical systems $(X, T)$ with $h(T) = 0$, $f \in C(X)$ and $x \in X$?
- What happens if $\boldsymbol{u} = \boldsymbol{\lambda}$? Then

$$\begin{aligned}\frac{1}{N}\sum_{n\leqslant N} f(T^n x)\boldsymbol{\lambda}(n) &= \frac{1}{N}\sum_{n\leqslant N} f(T^n x)\left(\sum_{d^2|n}\boldsymbol{\mu}(n/d^2)\right)\\ &= \frac{1}{N}\sum_{n\leqslant N}\sum_{d^2|n}\boldsymbol{\mu}(n/d^2) f((T^{d^2})^{n/d^2}x)\\ &= \sum_{d\leqslant\sqrt{N}}\frac{1}{d^2}\cdot\frac{1}{N/d^2}\sum_{n\leqslant N/d^2}\boldsymbol{\mu}(n) f((T^{d^2})^n x),\end{aligned}$$

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- Liouville orthogonality implies Möbius orthogonality: use the identity $\mu(n) = \sum_{d^2|n} \mu(d)\lambda(n/d^2)$ or use dynamics (joinings of zero entropy systems remain of zero entropy):[1]
- $\frac{1}{N}\sum_{n\leqslant N} f(T^n x)\mu(n) = \frac{1}{N}\sum_{n\leqslant N} f(T^n x)\mu^2(n)\lambda(n) = \frac{1}{N}\sum_{n\leqslant N}(f\otimes Z_0)((T\times S)^n(x,\mu^2))\lambda(n)\to 0$; the point $(x,\mu^2)$ is completely deterministic (we use a theorem that Sarnak's conjecture is equivalent to the validity of (S) for each completely deterministic $x\in X$, El Abdalauoi, Kułaga-Przymus, L., de la Rue, 2014).
- But what to do with a general multiplicative $\boldsymbol{u}$? We assume that $\boldsymbol{u}$ is aperiodic (i.e. is orthogonal to all periodic sequences).
- In 2010 it was reasonable to think that (S) holds if $\mu$ is replaced by such a $\boldsymbol{u}$ because of Elliott's conjecture! We use here the famous implication: the Chowla conjecture implies Sarnak's conjecture (Sarnak 2010, Tao 2012).
- In 2015, Matomäki, Radziwiłł and Tao disproved Elliott's conjecture (in fact they disproved even the "Chowla conjecture of order 2" for an aperiodic $\boldsymbol{u}$).

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- *Topological dynamical system* is a pair $(X, T)$, where $X$ is a compact metric space and $T : X \to X$ is a homeomorphism.
- $A$ – compact alphabet (usually $A \subset \mathbb{D}$, where $\mathbb{D}$ is the unit disc); if $X \subset A^{\mathbb{Z}}$ is closed and $S$-invariant, where for $y \in X$, $(Sy)(n) := y(n+1)$, $n \in \mathbb{Z}$ (left shift), then $(X, S)$ is a (topological) dynamical system, so called *subshift*.
- If $\boldsymbol{u} \in A^{\mathbb{Z}}$, then $u$ determines a subshift $X_{\boldsymbol{u}} := \overline{\{S^n \boldsymbol{u} : n \in \mathbb{Z}\}}$.
- If $\boldsymbol{u} \in A^{\mathbb{N}}$ then we may extend it symmetrically $\boldsymbol{u}(-n) = \boldsymbol{u}(n)$ ($\boldsymbol{u}(0) \in A$). Then consider $X_{\boldsymbol{u}}$ ...
- Consider the Liouville function $\boldsymbol{\lambda}(n) = (-1)^{\Omega(n)}$ for $n \geqslant 1$; here $A = \{-1, 1\}$, and we set $\boldsymbol{\lambda}(-n) = \boldsymbol{\lambda}(n)$ and $\boldsymbol{\lambda}(0) = 1$; $(X_{\boldsymbol{\lambda}}, S)$ is the Liouville subshift ($X_{\boldsymbol{\lambda}} \subset \{-1, 1\}^{\mathbb{Z}}$).

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# Topological dynamics. Visible measures and (quasi-)generic points

- $M(X)$ - space of probabilistic Borel measures on a compact metric space $X$; $M(X)$ is compact in the weak$^*$ topology.
- $M^e(X,T) \subset M(X,T) \subset M(X)$: set of $T$-invariant measure (closed subspace) and ergodic measures (extremal points).
- $M(X) \ni \frac{1}{N_k}\sum_{n\leqslant N_k}\delta_{T^n x} \to \mu$, then $\mu \in M(X,T)$ (in particular, $M(X,T) \neq \emptyset$).
- A point $x \in X$ is called *generic* for $\mu \in M(X,T)$, if $\frac{1}{N}\sum_{n\leqslant N}\delta_{T^n x} \to \mu$, that is, for each $f \in C(X)$, we have

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- If the convergence takes place along a subsequence, $x$ is called *quasi-generic*.
- Each point is quasi-generic for a measure in $M(X,T)$.
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- $M(X) \ni \frac{1}{L_{N_k}} \sum_{n \leqslant N_k} \frac{1}{n} \delta_{T^n x} \to \mu$, (here $L_N = \sum_{n \leqslant N} \frac{1}{n}$) then $\mu \in M(X, T)$.
- $V^{\log}(x) := \{\mu \in M(X, T) :$ $x$ is logarithmically quasi-generic for $\mu\}$.
- Either $|V(x)| = 1$ or $V(x)$ is uncountable (and connected).
- Let $A \subset \mathbb{D}$ be compact and $\boldsymbol{u} \in A^{\mathbb{Z}}$. We can consider $V(\boldsymbol{u}) = V_S(\boldsymbol{u})$.

**Definition**

Given $\boldsymbol{u} \in A^{\mathbb{Z}}$, by a Furstenberg system of it, we mean a measure-theoretic system $(X_{\boldsymbol{u}}, \kappa, S)$, where $\kappa \in V(\boldsymbol{u})$ is arbitrary.

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# Chowla conjecture and Furstenberg systems

**Chowla conjecture (1965):**

$$\lim_{N\to\infty}\frac{1}{N}\sum_{1\leqslant n\leqslant N}\boldsymbol{\lambda}(n+a_1)\cdot\ldots\cdot\boldsymbol{\lambda}(n+a_k)=0$$

for any choice of $0\leqslant a_1<\ldots<a_k$, $k\geqslant 1$.

- It is equivalent to: $\boldsymbol{\lambda}$ is a generic point for the Bernoulli measure in $\{-1,1\}^{\mathbb{Z}}$, i.e. $V(\boldsymbol{\lambda})=\{B(1/2,1/2)\}$.
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# Chowla type conjecture in totally aperiodic case versus Furstenberg systems

- Recall: $\boldsymbol{u} : \mathbb{N} \to \mathbb{D}$ is *multiplicative* if $\boldsymbol{u}(mn) = \boldsymbol{u}(m)\boldsymbol{u}(n)$ whenever $(m, n) = 1$. If $\boldsymbol{u} : \mathbb{N} \to \mathbb{S}^1$ is *aperiodic* (i.e. its mean along any arithmetic sequence exists and equals zero), and all powers $\boldsymbol{u}^k$ ($k \geqslant 1$) are still aperiodic,[2] then the analog of Chowla conjecture for $\boldsymbol{u}$ becomes:

**Chowla type conjecture for $\boldsymbol{u} : \mathbb{N} \to \mathbb{S}^1$**

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leqslant N} \boldsymbol{u}^{r_1}(n+a_1)\cdot\ldots\cdot\boldsymbol{u}^{r_k}(n+a_k)\overline{\boldsymbol{u}^{s_1}(n+b_1)\cdot\ldots\cdot\boldsymbol{u}^{s_\ell}(n+b_\ell)}=0$$

for all powers $r_u, s_t \in \mathbb{N}$ and $\{a_1, \ldots, a_k\} \cap \{b_1, \ldots, b_\ell\} = \emptyset$.

- It is equivalent to: $\boldsymbol{u}$ is generic for Haar measure on $(\mathbb{S}^1)^{\mathbb{Z}}$, i.e. $V(\boldsymbol{u}) = \{Leb_{\mathbb{S}^1}^{\otimes\mathbb{Z}}\}$.
- It implies $X_{\boldsymbol{u}} = (\mathbb{S}^1)^{\mathbb{Z}}$ and the topological entropy is infinite.

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$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leqslant N} \boldsymbol{u}^{r_1}(n+a_1)\cdot\ldots\cdot\boldsymbol{u}^{r_k}(n+a_k)\overline{\boldsymbol{u}^{s_1}(n+b_1)\cdot\ldots\cdot\boldsymbol{u}^{s_\ell}(n+b_\ell)}=0$$

for all powers $r_u, s_t \in \mathbb{N}$ and $\{a_1,\ldots,a_k\} \cap \{b_1,\ldots,b_\ell\} = \emptyset$.

- It is equivalent to: $\boldsymbol{u}$ is generic for Haar measure on $(\mathbb{S}^1)^{\mathbb{Z}}$, i.e. $V(\boldsymbol{u}) = \{Leb_{\mathbb{S}^1}^{\otimes\mathbb{Z}}\}$.
- It implies $X_{\boldsymbol{u}} = (\mathbb{S}^1)^{\mathbb{Z}}$ and the topological entropy is infinite.

[2] $\boldsymbol{u}$ is *totally* aperiodic.

# Some words on general conjectures

- Elliott's conjecture[3] (1992).
- Disproved by Matomäki, Radziwiłł and Tao in 2015 (2-Chowla fails...).
- Corrected Elliott's conjecture (by Matomäki, Radziwiłł, Tao) for so called strongly aperiodic functions.

**Questions:**

What about Furstenberg systems of aperiodic multiplicative functions? What are possible entropies? Is topological entropy positive? Does Chowla hold along a subsequence?

- Asked as Problem 7.3 during the *Sarnak's Conjecture* workshop at AIM in December 2018.

[3] In particular, it implies that Chowla holds for (totally) aperiodic multiplicative functions. Furstenberg systems for even strongly aperiodic but not totally aperiodic complex valued multiplicative functions, less clear: $n^{it}\boldsymbol{\lambda}(n)$.

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# Some words on general conjectures

**Frantzikinakis-Host's conjectures (2017, 2018)**

(A) Each multiplicative function $\boldsymbol{u} : \mathbb{N} \to [-1, 1]$ has a unique (logarithmic) Furstenberg system. The system is isomorphic to the direct product of an ergodic procyclic system and a Bernoulli system.

(B) For each multiplicative function $\boldsymbol{u} : \mathbb{N} \to \{-1, 1\}$ the unique (logarithmic) Furstenberg system is either Bernoulli or an ergodic odometer. It is Bernoulli if and only if $\boldsymbol{u}$ is aperiodic.

(C) All (logarithmic) Furstenberg systems of any multiplicative function $\boldsymbol{u} : \mathbb{N} \to \mathbb{S}^1$ have ergodic components isomorphic to direct products of procyclic systems and Bernoulli systems.

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# Archimedean characters

- *Archimedean character* is a completely multiplicative function $n \mapsto n^{it}$ with some fixed $t \in \mathbb{R}$.
- Archimedean characters have no mean: $\frac{1}{N}\sum_{1\leqslant n\leqslant N} n^{it} = \frac{N^{it}}{1+it} + o(1)$, in particular, they are not aperiodic.
- $\left|(n+1)^{it} - n^{it}\right| = \left|e^{it\log(1+1/n)} - 1\right| \xrightarrow[n\to\infty]{} 0,$
- Archimedean characters are slowly varying arithmetic functions $\boldsymbol{u}$:

$$\boldsymbol{u}(n+1) - \boldsymbol{u}(n) \xrightarrow[n\to\infty]{} 0.$$

- In particular, they are mean slowly varying function: $\frac{1}{N}\sum_{n\leqslant N} |\boldsymbol{u}(n+1) - \boldsymbol{u}(n)| \xrightarrow[N\to\infty]{} 0.$
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**Proposition**

The arithmetic function $\boldsymbol{u} : \mathbb{N} \to \mathbb{D}$ is mean slowly varying if and only if all Furstenberg systems of $\boldsymbol{u}$ are measure-theoretically isomorphic to the action of the identity[a] on some probability space.

[a] That is, they are identities ...

- Let $\frac{1}{N_m} \sum_{n \leqslant N_m} \delta_{S^n \boldsymbol{u}} \to \nu$, $Z_0 : X_{\boldsymbol{u}} \to \mathbb{D}$, $Z_0(y) := y(0)$, $Z_n := Z_0 \circ S^n$.
- Let $\boldsymbol{u}$ be mean slowly varying.
- $\mathbb{E}_\nu [|Z_1 - Z_0|] = \lim_{m\to\infty} \frac{1}{N_m} \sum_{n \leqslant N_m} |\boldsymbol{u}(n+1) - \boldsymbol{u}(n)| = 0$.
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# Furstenberg systems of mean slowly varying functions

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# Furstenberg systems of Archimedean characters and slightly beyond

- $\boldsymbol{u}(n) = n^i$, $n \geqslant 1$;
- Let $g(e^{2\pi i x}) = \frac{2\pi e^{2\pi x}}{e^{2\pi}-1}$ $(x \in [0,1))$.

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We have $X_{\boldsymbol{u}} = \{(\ldots, z, z, \ldots) : z \in \mathbb{S}^1\} \cup \{S^n \boldsymbol{u} : n \in \mathbb{N}\}$.[a] The family of Furstenberg systems of $\boldsymbol{u}(n) = n^i$ consists of uncountably many different systems given by all rotations of the measure $g(z)dz$. All of them are isomorphic to the identity on the circle with Lebesgue measure.

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- In fact, each invariant measure for an **arbitrary** slowly varying function yields identity (topological entropy is zero).
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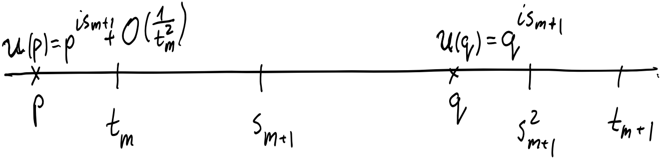
# Locally Archimedean characters. The MRT class

**Definition (Matomäki, Radziwiłł, Tao; 2015)**

A completely multiplicative function $\boldsymbol{u} : \mathbb{N} \to \mathbb{S}^1$ belongs to the MRT class if there exist two increasing sequences of integers $(t_m)$ and $(s_m)$ such that, for each $m \geqslant 1$, we have the following properties:

- $t_m < s_{m+1} < s_{m+1}^2 \leqslant t_{m+1}$,
- for each prime $p \in (t_m, t_{m+1}]$, $\boldsymbol{u}(p) = p^{is_{m+1}}$,
- for each prime $p \leqslant t_m$, $\left|\boldsymbol{u}(p) - p^{is_{m+1}}\right| < \dfrac{1}{t_m^2}$.


$u(p) = p^{is_{m+1}} + O\left(\frac{1}{t_m^2}\right)$
$u(q) = q^{is_{m+1}}$
$p$
$t_m$
$s_{m+1}$
$q$
$s_{m+1}^2$
$t_{m+1}$

# How to get functions in MRT?

- We have to define $\boldsymbol{u}(p)$ for each prime $p$ and to construct the sequences $(t_m)$ and $(s_m)$ (done inductively).
- Start by choosing an integer $t_1 \in \mathbb{N}$ and set, for each prime $p \leqslant t_1$, $\boldsymbol{u}(p) := 1$.
- Assume that for some $m \geqslant 1$ we have already defined $t_m$ and $\boldsymbol{u}(p)$ for each $p \leqslant t_m$.
- In the Cartesian product $\prod_{p \leqslant t_m} \mathbb{S}^1$, we consider the sequence of points

$$\left( \left( p^{is} \right)_{p \leqslant t_m} \right)_{s \in \mathbb{N}}.$$

- Since the numbers $\log p$, $p \leqslant t_m$, are linearly independent over the integers, this sequence is dense in $\prod_{p \leqslant t_m} \mathbb{S}^1$. Thus, we can choose $s_{m+1} > t_m$ so that $\boldsymbol{u}(p) = p^{is_{m+1}} + O(\frac{1}{t_m^2})$ is satisfied.

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- The growth of $s_{m+1}/t_m$ is necessarily superpolynomial: for each $\beta > 0$, $t_m < s_{m+1}^{\beta}$ for $m$ large enough.
- It is shown by Matomäki, Radziwiłł and Tao that once $s_{m+1} > e^{t_m}$ for $m \geqslant 1$, the resulting $\boldsymbol{u}$ is aperiodic.[4]

[4] In fact, totally aperiodic.

# Basics of the construction

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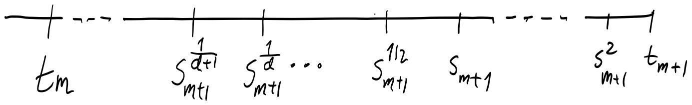
$t_m$
$s_{m+1}^{\frac{1}{d+1}}$
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...
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$s_{m+1}$
$s_{m+1}^2$
$t_{m+1}$

**Main Theorem (Gomilko, L., de la Rue, 2020)**

Let $\boldsymbol{u}$ be in the MRT class. Then, for each $d \geqslant 0$, there is a Furstenberg system $(X_{\boldsymbol{u}}, \nu_d, S)$ of $\boldsymbol{u}$ which is measure-theoretically isomorphic[a] to the unipotent system

$$A_d : (x_d, x_{d-1}, \ldots, x_0) \mapsto (x_d, x_{d-1} + x_d, \ldots, x_0 + x_1)$$

on $\mathbb{T}^{d+1}$ equipped with the $(d+1)$-dimensional Lebesgue measure. Furthermore, the Bernoulli shift $\Big((\mathbb{S}^1)^{\mathbb{Z}}, (Leb_{\mathbb{S}^1})^{\otimes\mathbb{Z}}, S\Big)$ is also a Furstenberg system of $\boldsymbol{u}$, i.e. the Chowla conjecture holds for $\boldsymbol{u}$ along a subsequence. In particular, $X_{\boldsymbol{u}} = (\mathbb{S}^1)^{\mathbb{Z}}$ and $h_{\mathrm{top}}(X_{\boldsymbol{u}}, S) = \infty$.

[a] Under the isomorphism, the stationary process $(Z_n)_{n\in\mathbb{Z}}$ corresponds to $(F_d \circ A_d^n)_{n\in\mathbb{Z}}$, where $F_d(x_d, \ldots, x_0) = e^{2\pi i x_0}$.

- Hence, the family $\{\nu_d : d \geqslant 0\}$ makes the process(es) $(Z_n)$ more and more independent. Since $V_S(\boldsymbol{u})$ is closed,

$$\lim_{d\to\infty} \nu_d = (Leb_{\mathbb{S}_1})^{\mathbb{Z}} \in V_S(\boldsymbol{u}).$$

- Haar measure on $(\mathbb{S}^1)^{\mathbb{Z}}$ is a Furstenberg system of $\boldsymbol{u}$.[5]
- $X_{\boldsymbol{u}} = (\mathbb{S}^1)^{\mathbb{Z}}$. (NOTE THAT it means that, for each $\varepsilon > 0$, ALL $\varepsilon$-configurations appear in $\boldsymbol{u}$).[6]
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# Discussion...

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[6] More than that: For each configuration, its $\varepsilon$-nbhd appears with positive upper density on $\boldsymbol{u}$.

# Other properties of members of MRT class

- If $\boldsymbol{u}$ is an MRT function then it does not satisfy Sarnak's conjecture (a zero entropy system which correlates with $\boldsymbol{u}$ is close to a one given by a slowly varying function).
- $\boldsymbol{u}$ does not satisfy the "zero mean property on typical short interval"; cf. Matomäki-Radziwiłł theorem for strongly aperiodic functions

$$\lim_{M,H\to\infty, H=o(M)} \frac{1}{M} \sum_{1\leqslant m\leqslant M} \left| \frac{1}{H} \sum_{0\leqslant h<H} \boldsymbol{u}(m+h) \right| = 0.$$

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# Logarithmic Chowla conjecture

$\boldsymbol{u}$ satisfies logarithmic Chowla conjecture along a subsequence - proof:

- $E_N^{\log}(\boldsymbol{u}) = \frac{1}{L_N}\sum_{1\leqslant n\leqslant N-1}\frac{1}{n+1}E_n(\boldsymbol{u}) + \frac{1}{L_N}E_N(\boldsymbol{u})$, where $E_N^{\log}(\boldsymbol{u}) := \frac{1}{L_N}\sum_{1\leqslant n\leqslant N}\frac{1}{n}\delta_{S^{n-1}\boldsymbol{u}}$
- If we fix $\frac{1}{d+1} < \beta_d < \beta'_d < \frac{1}{d}$, $1 < \beta_0 < \beta'_0 < 2$ then $E_N(\boldsymbol{u})$ is **uniformly** close to $\nu_d$ for $s_{m+1}^{\beta_d} \leqslant N \leqslant s_{m+1}^{\beta'_d}$ (and $m$ large enough).
- For parameters $\varepsilon > 0$, $1 \leqslant D_1 < D_2$ we show that $D_1(1-\varepsilon)\sum_{D_1\leqslant d\leqslant D_2}\left(\frac{1}{d} - \frac{1}{d+1}\right)\nu_d + \alpha\rho$ is a logarithmic Furstenberg system.
- $(*)$ $D_1\sum_{d\geqslant D_1}\left(\frac{1}{d} - \frac{1}{d+1}\right)\nu_d \in V_S(\boldsymbol{u})$.
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Question: Can we find a Furstenberg system for some $\boldsymbol{u} \in \mathrm{MRT}$ which is a (non-trivial) direct product of a Bernoulli and a nilpotent system? (N. Frantzikinakis, F. Richter).

Thank you!