THE SIXTH POWER MOMENT AND THE ZERO-DENSITY ESTIMATES OF THE RIEMANN ZETA FUNCTION

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1. ASYMPTOTICS FOR POWER MOMENTS

The power moments

(1) $$M_k(T) = \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt$$

became the center of recent studies in the Riemann zeta function after their connection with the RMT (Random Matrix Theory) was established in 2000 by KEATING and SNAITH. Together with CONREY, GHOSH, FARMER and others they built a great industry in this area.

From early years the power moments $M_k(T)$ are being used for counting zeros off the critical line;

$$N(\alpha,T) = \#\{\rho=\beta+i\gamma;\ \zeta(\rho)=0,\ \alpha\le\beta<1,\ |\gamma|\le T\},$$

see H.L. Montgomery, LNM 227.

The Riemann Hypothesis implies the Lindelöf Hypothesis

(2) $$\zeta(\tfrac{1}{2}+it) \ll t^{\varepsilon}, \qquad t\geq 2.$$

Hence

(3) $$M_k(T) \ll T^{1+\varepsilon}, \qquad T \geq 2.$$

In fact, both estimates are equivalent.

It is more ambitious to prove the asymptotic formula

(4) $$M_k(T) \sim c_k T (\log T)^{k^2}$$

for every fixed positive integer k. The RH does not help entirely, but we have the bounds

$$T(\log T)^{k^2} \ll M_k(T) \ll T(\log T)^{k^2}.$$

RAMACHANDRA, 1980 HARPER + RH, 2013

So far we know unconditionally that (4) holds for

$k=1$ with $c_1 = 1$ (Hardy-Littlewood, 1922),
$k=2$ with $c_2 = 1/2\pi^2$ (Ingham, 1926).

For $k=3$, even the bound (3), that is

(5) $$\int_0^T |\zeta(\tfrac{1}{2}+it)|^6 \, dt \ll T^{1+\varepsilon}$$

is not yet provable by current technology.

Interesting attempts have been made to evaluate

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4 \Big|\sum_n a_n n^{it}\Big|^2 dt$$

for arbitrary coefficients a_n, $n \le T^\theta$, θ as large as possible.

CONREY and GHOSH (1998) developed a heuristic method to derive (4) for $k=3,4$, and later together with GONEK and FARMER they developed a far reaching heuristic (called the RECIPE) leading to (4) for every $k \geq 1$. Independently KEATING and SNAITH derived the same asymptotics via modelling the zeta function by the characteristic polynomials of random matrices (GUE).

CONJECTURE. The asymptotic formula (4) holds with $c_k = g_k a_k$, $g_k = G(1+k)^2/G(1+2k)$, $G(n) = 1!\,2!\ldots(n-2)!$,

$$a_k = \prod_p \left(1-\frac{1}{p}\right)^{(k-1)^2} \sum_{0\le l<k} \binom{k-1}{l}^2 p^{-l}.$$

In particular

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^6 dt \sim g_3 a_3 T (\log T)^9 \tag{6}$$

with

$$g_3 = \frac{42}{9!}, \qquad a_3 = \prod_p \left(1-\frac{1}{p}\right)^4 \left(1+\frac{4}{p}+\frac{1}{p^2}\right).$$

2. THE SIXTH POWER MOMENT FOR PARTIAL SUMS

Our goal is to estimate

$$(7) \qquad \mathcal{M}_6(T,M) = \int_0^T \Big| \sum_m m^{-\frac{1}{2}+it} \Big|^6 dt$$

where m runs over a segment $M < m \leq M' < 2M$ with

$$T^{\frac{1}{3}} < M < T^{\frac{1}{2}}.$$

Every moment $M_k(T)$ amounts to correlation of divisor type functions. In the case of $\mathcal{M}_6(T,M)$ we need to count solutions of

$$m_1 m_2 m_3 - m_4 m_5 m_6 = h$$

for

$$h \ll H = M^3 T^{-1}, \qquad 1 < H < T^{\frac{1}{2}}.$$

Estimating trivially one gets

$$(8) \qquad \mathcal{M}_6(T,M) \ll H T^{1+\varepsilon}.$$

We hope to get

$$(9) \qquad \mathcal{M}_6(T,M) \ll H^{\frac{1}{2}} T^{1+\varepsilon}.$$

3. THE ZERO-DENSITY BOUNDS

$N(\alpha, T) = 0$, if $\frac{1}{2} < \alpha < 1$, this is the RH

$N(\alpha, T) \ll T^{c(\alpha)(1-\alpha)+\varepsilon}$,

$c(\alpha) = 2$, the Density Conjecture.

If $c(\alpha) \leq c$ for all α, then

(10) $$\sum_{x < p \leq x+y} \log p \sim y$$

for $y > x^{\theta+\varepsilon}$, $\theta = 1 - 1/c$ (G. Hoheisel, 1930)

$$RH \Rightarrow LH \Rightarrow DC \Rightarrow \theta = 1/2.$$

Classical estimates

$c(\alpha) = 4\alpha$, Carlson, 1920,

$c(\alpha) = \frac{3}{2-\alpha}$, Ingham, 1940.
$c = 3$, $\theta = 2/3$

Crucial improvements of $c(\alpha)$ were obtained in the segment $\frac{3}{4} < \alpha < 1$ by HALÁSZ method, 1968.

$c(\alpha) = \frac{2}{\alpha}$, Montgomery, 1969, $c = 5/2$, $\theta = 3/5$

$c(\alpha) = \frac{3}{3\alpha - 1}$, Huxley, 1972. $c = 12/5$, $\theta = 7/12$

The sixth power moment estimate (5) yields

$c(\alpha) = \frac{8}{5 - 2\alpha}$, by JUTILA's method. $c = 16/7$, $\theta = 9/16$

The same $c(\alpha)$ follows from (9), again by Jutila's method.

HALÁSZ and TURÁN (1969) showed that the Lindelöf Hypothesis implies

$$N(\alpha, T) \ll T^{\varepsilon}, \quad \text{if } \frac{3}{4} < \alpha \leq 1.$$

4. DIRICHLET POLYNOMIALS AND KLOOSTERMAN SUMS

$$\mathcal{A}(s) = \sum_{m \sim M} a_m m^{-s}, \qquad \mathcal{B} = \sum_{m \sim M} b_m m^{-s},$$

$$I = \int K\left(\frac{t}{T}\right) \mathcal{A}\left(\tfrac{1}{2}+it\right) \mathcal{B}\left(\tfrac{1}{2}-it\right) dt$$

$$= \sum_m \sum_n a_m b_n (mn)^{-\frac{1}{2}} I\left(\frac{m}{n}\right),$$

$$I\left(\frac{m}{n}\right) = \int K\left(\frac{t}{T}\right)\left(\frac{m}{n}\right)^{it} dt.$$

$K(u)$ smooth, compactly supported on $\mathbb{R}^+$.

The resulting equations

$$m - n = h, \qquad h \ll H = MT^{-1}$$

are treated by Kloostermon Circle Method, alternatively by Delta Method.

PROPOSITION. Let $f(y)$ be continuous on $\mathbb{R}$ with the Fourier transform $\hat{f}(z) \ll (1+|z|)^{-2}$. For every integer k we have

$$f(k) = 2\mathrm{Re} \int_0^1 \sum_{\substack{ad-bc=1 \\ 1 \le c \le C < d \le c+C}} \sum \sum \sum (cd)^{-1} \hat{f}\left(\frac{a}{c} - \frac{x}{cd}\right) e\left(k\left(\frac{a}{c} - \frac{x}{cd}\right)\right) dx .$$

EXAMPLES.

$a_m = \tau(m)$, $b_m = \tau(m)$, GL_2 case,
$a_m = \tau_3(m)$, $b_m = \tau_3(m)$, GL_3 case,

$$a_m = \sum_{m_1 m_2 m_3 = m} f(m_1, m_2, m_3), \qquad b_n = \sum_{n_1 n_2 n_3 = n} g(n_1, n_2, n_3).$$

Applying Poisson's formula (in six variables) to the BILINEAR FORMS

$$V = \sum_m \sum_n a_m b_n e\left((m-n-h)\left(\frac{a}{c} - \frac{x}{cd}\right)\right)$$

we get DUAL BILINEAR FORMS

$$V^* = \sum_l \sum_{l'} \phi(l)\phi(l') S(\bar{a}, l; c) S(\bar{a}, l'; c)$$

where the coefficients are still of GL_3 type

(11) $$\phi(l) = \sum_{l_1 l_2 l_3 = l} \phi(l_1, l_2, l_3) \ll l^{\varepsilon}$$

and

(12) $$S(\bar{a}, l; c) = \sum_{x (\mathrm{mod}\, c)}^{*} e\left(\frac{\bar{a}\bar{x} + lx}{c}\right) \ll c^{\frac{1}{2}} \tau(c)$$

is the Kloosterman sum.

5. CONJECTURES ON KLOOSTERMAN SUMS

CONJECTURE A. Let $(a,c)=1$ and $\phi(l)$ be of type (11). Then

$$\sum_{l\sim L} \phi(l)\, S(\bar{a}, l; c) \ll (c+L)(cL)^{\varepsilon}.$$

CONJECTURE B. Let $1 \le a \le C$ and $\phi(l)$ be of type (11). Then

$$\sum_{\substack{c\sim C \\ (c,a)=1}} \Big|\sum_{\substack{l\sim L \\ (l,c)=1}} \phi(l)\, S(\bar{a}, l; c)\Big|^2 \ll (C+L)^2 (CL)^{1+\varepsilon}.$$

CONJECTURE C. Let $C^{1/3} \le A \le C$. For any complex numbers $\phi(l) \ll 1$ we have

$$\sum_{c\sim C} \sum_{\substack{a\sim A \\ (a,c)=1}} \Big|\sum_{\substack{l\sim C \\ (l,c)=1}} \phi(l)\, S(\bar{a}, l; c)\Big|^2 \ll A^{\frac{1}{2}} C^{\frac{7}{2}+\varepsilon}.$$

CONJECTURE D. For every complex numbers $\phi(l) \ll 1$ we have

$$\sum_{c\sim C} \sideset{}{^*}\sum_{a \pmod c} \Big|\sideset{}{^*}\sum_{l\sim C} \phi(l)\, S(\bar{a}, l; c)\Big|^4 \ll C^{6+\varepsilon}.$$

6. EPILOGUE

Each one of the four conjectures about the Kloosterman sum (12) is strong enough to give the bound (9) for $\mathcal{M}_6(T,M)$. Conjecture $\mathbb{C}$ seems to be provable by means of spectral theory of automorphic forms on GL_3.

We need to develop a spectral expansion of

$$(13) \qquad \sum_{(c,a)=1} S(\bar{a},m;c)\,S(\bar{a},n;c)\,W(c/C)$$

This sum over moduli c resemble the Rankin-Selberg convolution of the Fourier coefficients.

There are publications remotely related to (13). See the recent papers (and references therein) by

JACK BUTTCANE, On Sums of Hyper-Kloosterman Sums, arXiv 2205.14194, May 27, 2022,

VALENTIN BLOMER and JACK BUTTCANE, Global Decomposition of GL(3) Kloosterman Sums and the Spectral Large Sieve, J. Reine Angew. Math. 757 (2019), 51-88.

THANK YOU !