Expansion, divisibility, parity

(joint w/ Maks Radziwiłł)

Motivation

Estimating correlations — basic problem in analytic NT

$$\frac{1}{X}\sum_{n\le X} a(n)\, b(n+h)$$ as $X\to\infty$

"Fine multiplicative structure" of $\mathbb{Z}$ $\longrightarrow$ Liouville function

$$\lambda(n) := (-1)^{\Omega(n)}$$

where $\Omega(n)$ is the # of prime factors of n

How hard are these questions?

$$\frac{1}{X}\sum_{n\le X}\lambda(n)\lambda(n+1) = o(1)$$ is open and very hard (Chowla's conjecture in degree 2)

... whereas $\frac{1}{x}\sum_{n\le x}\lambda(n) = o(1) \iff$ Prime Number Theorem

2015 Matomäki - Radziwiłł

$\lambda(n)$ averages to 0 over most short intervals

$$\frac{1}{NH}\sum_{N<x\le 2N}\left|\sum_{x<n\le x+H}\lambda(n)\right| = o(1)$$

provided that $H\to\infty$ as $N\to\infty$

(was known for $H \geq N^{1/6}$)

Tao (using Matomäki - Radziwiłł)

(*) $$\frac{1}{\log x}\sum_{n\le x}\frac{\lambda(n)\lambda(n+1)}{n} = o(1)$$

his method gives the bound

$$\dots = O\left(\frac{1}{(\log\log\log\log x)^{\alpha}}\right),\ \alpha>0$$

First step of Tao's proof?

(*) reduces to $$\frac{1}{x\mathcal{L}}\sum_{n\le x}\lambda(n)\sum_{\substack{p\in\mathbf{P}\\ p\mid n}}\lambda(n+p) = o(1)$$

($\mathbf{P}$ some set of primes, $\mathcal{L} = \sum_{p\in\mathbf{P}} 1/p$)

So, question: how to replace $p|n$ by weight $1/p$

Tao's way: $\exists$ set $\mathbf{P}$ for which this makes little difference

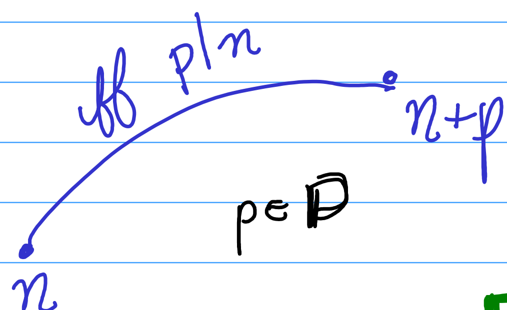
(Why? entropy is additive (exhaustion of mutual information)

Another way?

A prime divisibility graph

vertices $V = \{N+1, \dots, 2N\}$

edges $E = \{\{n, n+p\},\ p \in \mathbf{P},\ p|n,\ n, n+p \in V\}$



Γ

equivalent to graph considered by M-R-T II, Tao

"it may be possible to estimate expressions [...] by establishing some sort of expander graph property." "Unfortunately we were unable to establish such an expansion property, as the [...] standard methods of establishing expansion [do not] work."

(some sort of)
Expander graph property?

= property of spectrum of an linear operator on functions $f : V \to \mathbb{C}$

$\approx$ "random walks equidistribute quickly"

What operator?

Edges are short; aim for "strong local expansion a.e."

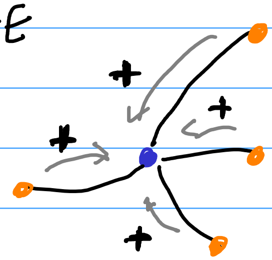
What bound do we want?

eigenvalues = o(average degree) $\approx L = \sum_{p \in P} 1/p$

Adjacency operator Ad_Γ

$$(Ad\, f)(v) = \sum_{w : \{v,w\} \in E} f(w)$$

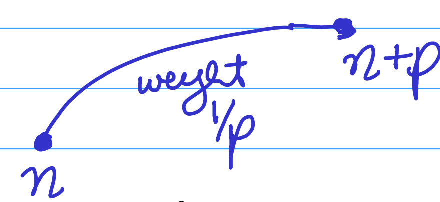
$Ad\, f : V \to \mathbb{C}$



Adjacency operator $Ad_{\Gamma'}$:

same, but for graph

Γ'



$p \in P$

"naive model"

We define

$$A := \mathrm{Ad}_\Gamma - \mathrm{Ad}_{\Gamma'}$$

Must still exclude some vertices (e.g. those of very high degree)

So, we'll work with a restriction $A|_X$ $X \subset V$ with small complement $V \setminus X$

MAIN THM. Let $\mathbf{P} \subset [H_0, H]$ a set of primes

A as above

$$\mathcal{L} = \sum_{p \in \mathbf{P}}{}' \, 1/p$$

Assume $(\log H_0) \geq (\log H)^{2/3+\varepsilon}$

$$H \leq e^{\sqrt{(\log N)/\mathcal{L}}}$$

Then

$\exists\, X \subset V$ with $|V \setminus X| \leq e^{-1000\mathcal{L}} N$

s.t.

every eigenvalue of $A|_X$ is $O(\sqrt{\mathcal{L}})$

Trivial bound: $O(\mathcal{L})$

So this is (so to speak) O(Ramanujan graph)

theoretical optimum

Cor.
$$\frac{1}{\log x}\sum_{n\le x}\frac{\lambda(n)\lambda(n+1)}{n} = O\left(\frac{1}{\sqrt{\log\log x}}\right)$$

or could be made to have

whereas Tao had $(\log\log\log\log x)^{-\alpha}$
and Tao-Teräväinen had $(\log\log\log x)^{-\alpha}$ $\alpha>0$

stronger Cor.
$$\frac{1}{x}\sum_{x<n\le 2x}\lambda(n)\lambda(n+1) = o(1)$$
at almost all scales, with stronger bounds than in Tao-Teräväinen

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Stronger Cor: For $I_1=I_1(x)$, $I_2=I_2(x)$ intervals,
$$\frac{1}{\log x}\sum_{\frac{x}{w}<n\le x}\frac{\lambda(n)\lambda(n+1)}{n} = O_\varepsilon\left(\frac{\sqrt{s_1 s_2}}{\log\log x}\right)$$
where $s_i=\min(|I_i|,\sqrt{\log\log x})$

Cor For any k,
$$\frac{1}{x}\sum_{\substack{x<n\le 2x\\ \Omega(n)=k}}\lambda(n+1) = O\left(\frac{1}{(\log\log x)^{\frac{3}{4}}}\right)$$
at almost all scales

Main ideas of proof

Step 0: $\exists$ a large eigenvalue $\Longrightarrow$ $\exists$ many large eigenvalues (because locality)

$\Longrightarrow$ trace $\mathrm{Tr}\, A_{|X}^{2k}$ is large

(even for moderate k)

Step 1: $\mathrm{Tr}(A_{|X}^{2k})$ = sum over closed paths (in $\Gamma - \Gamma'$) of length $2k$.

$\uparrow$ edge weights in Γ (0 or 1)

minus edge weights in Γ' ($1/p$)

cancellation when many primes appear exactly once each.

So, our task: count paths

n, p_1, $n_1 = n + \varepsilon_1 p_1$, p_2, $n_2 = n + \varepsilon_1 p_1 + \varepsilon_2 p_2$, p_3, p_i, $n + \varepsilon_1 p_1 + \varepsilon_2 p_2 + \dots + \varepsilon_i p_i$, p_{2k}

where almost every prime appears more than once.

$\varepsilon_i = \pm 1$

Weight? $\prod_{p_i \nmid n_i} 1/p_i$

Step 2:

Avoid early recurrences

by removing from X all $n \in N$ that cause them

That is, ($p_i \cdots p_j \cancel{\times} p_i \cdots p_{i'} (= p_i)$ with $i' - i$ small)

That's a small set: easy

That set is well-distributed: hard!

(so ~~that~~ we still get cancellation)

develop generalized Brun's sieve for conditions mod composites;

prevent combinatorial explosion using Rota's cross-cut thm

MAIN PART OF PROOF:

Bound # of closed paths with no early recurrences

and few non-repeated primes,

and small weight if many i are "bad" (i.e., $p_i \nmid n_i$)

If $p_i = p_{i'}$ and i, i' are "good", then clearly

$$p_i \mid \varepsilon_{i+1} p_{i+1} + \cdots + \varepsilon_{i'} p_{i'}$$

(system of divisibility relations)

We consider the shape of a walk

$\vec{\sigma} = \{-1, 1\}^{2k}$

shape: $(\sim, \vec{\sigma})$

equivalence relation on $\{1, \dots, 2k\}$

We will sum over all walks of that shape

$\mathbf{1} \subset \{1, \dots, 2k\}$ "lit" indices $\leftarrow p_i \mid n_i$

if $1 \sim 4$ and $1, 4 \in \mathbf{1}$:

in the example

$p_1 \mid n + p_1$, $p_1 = p_4 \mid n + p_1 - p_2 + p_3 + p_1$ $\implies$ $p_1 \mid p_2 + p_3$

if 1 or $4 \notin \mathbf{1}$:

pay penalty $\frac{1}{p_1}$ (or $\frac{1}{p_1^2}$)

$$\begin{array}{c|l} p_{[1]} & a_{11} p_{[1]} + a_{12} p_{[2]} + \dots \\ p_{[1]} & \\ p_{[2]} & M = (a_{ij}) \end{array}$$

Idea: we will usually find in M a submatrix of large rank r with disjoint row and columns indices

geometry of #s

few walks of that shape

Example:

$n \to n + p_1$

$\downarrow$

$n + p_1 - p_2$

$\downarrow$

$n + p_1 - p_2 + p_3$

$\downarrow$

$n + p_1 - p_2 + p_3 + p_1$

$\downarrow$

$n + 2p_1 - p_2 + p_3 + p_2$

$\downarrow$

$n + 2p_1 + p_3 - p_4 = n$

has shape $(\sim, \vec{\sigma})$:

$\vec{\sigma} = 1, -1, 1, 1, 1, -1$

equiv classes $i \sim j$ iff $p_i = p_j$

Sketch of proof

Let $\sim$ be an equiv. relation on $\{1, \dots, 2k\}$, $\vec{\sigma}$ a tuple $\in \{-1,1\}^{2k}$

$(\sim, \vec{\sigma})$ induces a word

The indices <u>not</u> surviving reductions are colored yellow

graph $G_\sim :=$ vertices = non-yellow equivalence classes

edges: between $[i_1]$ and $[i_2]$

s.t. every $i_1 < j < i_2$ or $i_2 < j < i_1$ is yellow

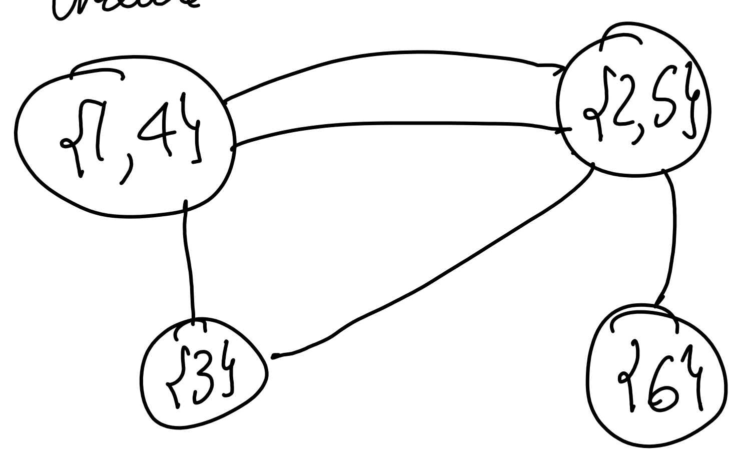
Example:

$\sim = \{\{1,4\}, \{2,5\}, \{3\}, \{6\}\}$

$\vec{\sigma} = (1, -1, 1, 1, 1, -1)$

$x_{[1]} x_{[2]}^{-1} x_{[3]} x_{[1]} x_{[2]} x_{[6]}^{-1}$

induce



Lemma Let $(\sim, \vec{\sigma})$ be a shape of length $2k$

Let $G = G_\sim$.

Partition non-yellow vertices of $G_\sim$ into red and blue so that $G|_{blue}$ is connected

Let $x_{[j]}$ a formal variable for each red $[j]$

$$v(i) := \sum_{\substack{j<i \\ [j]\in \text{red}}} \sigma_j \, x_{[j]} \qquad \text{for } 1 \le i \le 2k$$

Then the space V spanned by $v(i_2) - v(i_1)$ with $[i_1] = [i_2] \in$ blue

equals the space W spanned by $v(i_2) - v(i_1)$ with $[i_1] = [i_2] \in$ blue

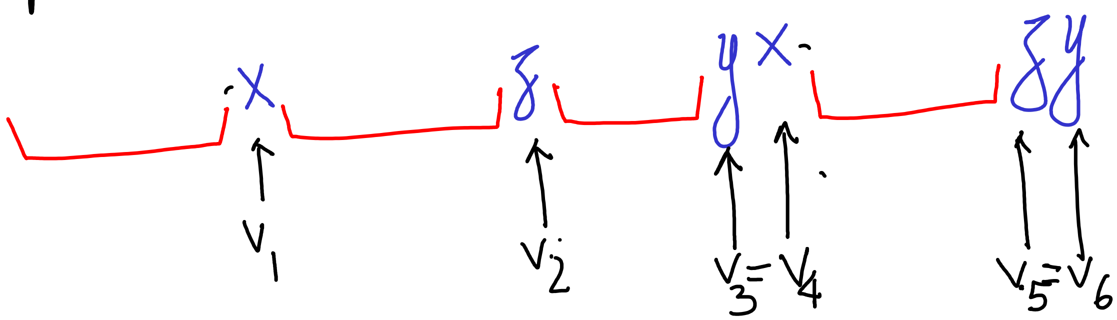
Easy linear algebra Lemma

Let A be an $n \times m$ matrix s.t.

- every row has at least one non-zero entry
- no column has $> k$ non-0 entries

Then $\operatorname{rank} A \ge \frac{n}{k}$

Example



because g_ν blue is connected

rows in M:

P_x | linear combination of red variables between first and second x $+ C$

$= v_4 - v_1$

$$V = \langle v_4 - v_1, v_6 - v_3, v_5 - v_2 \rangle$$
$$\|$$
$$W = \langle v_2 - v_1, v_3 - v_2, \overset{=0}{v_4 - v_3}, v_5 - v_4, \overset{=0}{v_6 - v_5} \rangle$$

Prop Let $(\nu, \vec{\sigma}), g_\nu, V$ be as before. Let $(\nu', \vec{\sigma}')$ be the reduced shape.
Assume $< k$ disjoint revenants in $(\nu', \vec{\sigma}')$ (i.e. can't have
$1 \le i_1 < i_1' \le i_2 < i_2' < \cdots \le i_r < i_r' \le 2k'$
with $i_r \sim i_r'$ $\forall r$
$\exists\, i_r < j < i_r'$ with $j \not\sim i_r$

Then $\dim W \geq \frac{s}{k} - 1$

where $s = \#$ of indices $1 \le j \le 2k'$ s.t.
j is blue and $j+1$ is red

Idea of Pf

s counts red blocks

$<k$ disjoint revenants $\longrightarrow$ $<k$ of these blocks have trivial sums

$\downarrow$

each red letter appears in $<k$ blocks

So: use easy linear algebra lemma

So: want to choose blue $\subset V$ so that $G|_{\text{blue}}$ is connected and s is as large as possible

Draw an arrow from $[i]'$ to $[i+1]'$ in G for each $1 \le i \le 2k'$ and from $[2k']$ to $[1']$ (draw each arrow only once)

There's an arrow into every vertex

For $S \subset V$, $\vec{\partial} S := \{ v \in V \setminus S : \exists\, v' \in S \quad v' \longrightarrow v \}$. Then $s \geq |\vec{\partial} S| - 1$.

So choose blue $\subset V$ so that G_{blue} is connected & $\vec{\partial}$ blue is large. How?

How to choose connected blue $\subset V$

s.t. $\vec{\partial}$ blue is large

Graph theory result

(essentially Kleitman-West '91;
followed Storer '91,
Payan-Tchuente-Xuong '84,
Griggs-Kleitman-Shastri '89;
see also Gravin '11,
Bankevich-Karpov '12, Karpov '13)

For any connected graph with $\geq n$ vertices of degree >2,

$\exists$ spanning tree $\geq \frac{n}{4}+2$ leaves. //

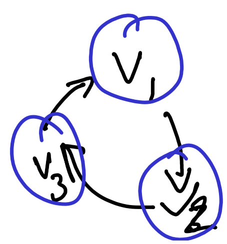
The undirected boundary of $\underbrace{\text{the set of non-leaves (internal vertices)}}_{\text{connected!!!}}$ is the set of leaves

directed boundary $\vec{\partial}$:

(0) put every internal vertex in blue

(1) if $\bullet$ non-leaf $\to$ leaf v, put v in red

(2) other leaves fall into cycles



odd

red

So:

- $G|_{\text{blue}}$ is connected
- $\vec{\partial}$ blue $> \frac{n}{12}+1$

Good!

But what happens if n is small,
i.e., if few vertices v in G have degree >2?

degree 1 means:

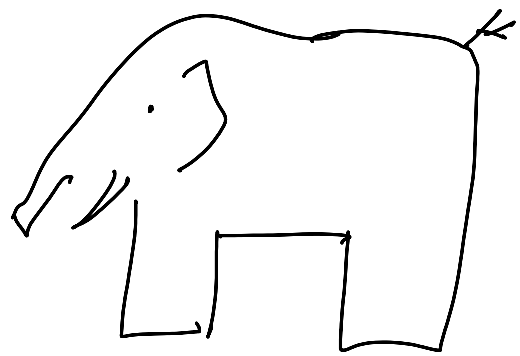
$w v w$ $\quad$ $w^{-1} v v v w^{-1}$

v always followed and preceded by w (or v)

degree 2:
same with w_1, w_2

→ words with few vertices of degree >2 can be described simply

→ few such words

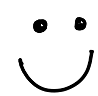


v

(meaning

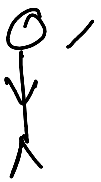
$O(1)^k \quad (2k)^{kn}$

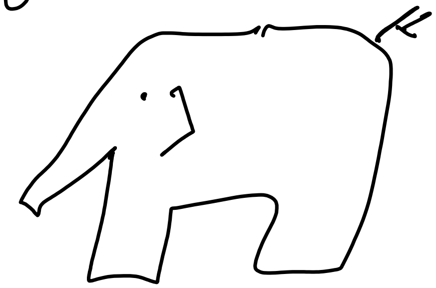
as opposed k^k)



What about the part of the walk that got reduced?

( ) ((( ))



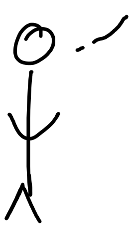


$\longrightarrow$ reconstructing $2m$ steps that were reduced has cost $O(KL)^m$.

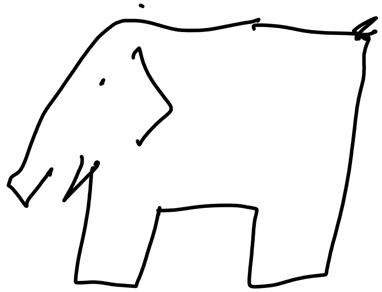
still have to describe equivalence class of each (

If new, good. If not:

$\cdots P_{[j]} \log_j P_{[j]} + \cdots \epsilon_{i-1} P_{[i-1]}$



It's the 5th one among the $[i,j]$'s that could reappear



... and there are only $\leq KL$ possibilities

Choose your parameters:

$k = c_0 \log H$, $\ell \simeq \frac{\log H_0}{\log L \log 2k}$, $\kappa \simeq \frac{\sqrt{\log H_0}}{\log 2k}$ $\longrightarrow$ Main Thm proved. //

What next?

- graphs with composite edges (perhaps edges with 2 prime factors) → without factors $< H_0$) → bounds $\frac{1}{(\log x)^c}$ rather than $\frac{1}{\sqrt{\log \log}}$?

$\Rightarrow$ a.e.

$\# n$ s.t. $\Omega(n) = k$
$\Omega(n+1) = k'$

k, k' popular

- higher-order Chowla?

unclear what to do hypergraph?