Fourier coefficients of Eisenstein series for $\mathrm{SL}(n, \mathbb{Z})$

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September 15, 2022

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(joint work with Eric Stade and Michael Woodbury)

Warm-Up: Eisenstein series for $SL(2, \mathbf{Z})$

Let $z = x + iy \in \mathfrak{h}^2$ and $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$. The Eisenstein series is defined by

$$E(z,s) := \sum_{(c,d)=1} \frac{y^s}{|cz+d|^s} = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}(2,\mathbb{Z})} \mathrm{Im}(\gamma z)^s$$

with Fourier expansion

$$E(z,s) = \underbrace{y^s + \phi(s) y^{1-s}}_{\text{constant term}} + \underbrace{\frac{2\pi^s}{\Gamma(s)\zeta(2s)}}_{\text{First Fourier Coeff.}} \sum_{n \neq 0} \underbrace{\sigma_{1-2s}(n) n^{s-\frac{1}{2}}}_{\text{Hecke eigenvalue}} W_n(z).$$

$$\sigma_s(n) = \sum_{d|n} d^s, \qquad W_n(z) = \frac{\sqrt{y}}{2} \int_0^\infty e^{-\pi|n|y(u+\frac{1}{u})} u^s \frac{du}{u} \cdot e^{2\pi i n x}$$

$$\phi(s) = \sqrt{\pi} \frac{\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\zeta(2s)}$$

OUTLINE OF THIS TALK

(1) Setting up basic notation.

(2) We define general Langlands Eisenstein series for $\mathrm{SL}(n, \mathbb{Z})$.

(3) The computation of the non-constant Hecke eigenvalues of Langlands Eisenstein series is derived in a simple manner.

(4) The value of the first coefficient for general Langlands Eisenstein series for $\mathrm{SL}(n, \mathbb{Z})$ is explicitly given.

Upper half plane $\mathfrak{h}^n$

Upper half plane: $\mathfrak{h}^n := GL(n,\mathbb{R}) / \left(O(n,\mathbb{R}) \cdot \mathbb{R}^\times\right)$

$$\boxed{g \in \mathfrak{h}^n \implies g = xy}$$

with

$$x \in U_n(\mathbb{R}) = \begin{pmatrix} 1 & & & x_{i,j} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix}$$

$$(x_{i,j} \in \mathbb{R}, \qquad y_i > 0)$$

Example: (classical upper half plane $\mathfrak{h}^2$)

$$\boxed{g = \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in \mathfrak{h}^2}$$

which is isomorphic to $\left\{x + iy \;\middle|\; x \in \mathbb{R},\ y > 0\right\}$.

Cusp forms for $GL(n, \mathbb{R})$

A cusp form is a complex valued function

$$\phi : \mathfrak{h}^n \to \mathbb{C}$$

satisfying the following conditions:

- $\boxed{\phi(\gamma g) \;=\; \phi(g),} \quad \big(\forall\, \gamma \in SL(n, \mathbb{Z}), \;\; g \in \mathfrak{h}^n\big);$
- *eigenfunction of invariant differential operators;*
- *has moderate growth;*
- *vanishes at the cusps.*

Invariant Differential Operators on $\mathfrak{h}^n$

Recall that $\mathfrak{h}^n$ can be identified with the set of matrices xy:

$$x \in U_n(\mathbb{R}) = \begin{pmatrix} 1 & & & x_{i,j} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix}$$

$$(x_{i,j} \in \mathbb{R}, \qquad y_i > 0).$$

The space $\mathcal{D}^n$

The space $\mathcal{D}^n$ consists of all polynomials (with complex coefficients) in the variables $\left\{ \frac{\partial}{\partial x_{i,j}}, \frac{\partial}{\partial y_k} \right\}$ which are invariant under $GL(n, \mathbb{R})$ transformations.

Example: (Laplacian)

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

Langlands Parameters

$$\boxed{\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \in \mathbb{C}^n,} \qquad (\alpha_1 + \alpha_2 + \cdots + \alpha_n = 0).$$

Langlands parameters can be used to construct a character of the torus which is an eigenfunction of all $\delta \in \mathcal{D}^n$.

Construction of a POWER FUNCTION = an eigenfunction of $\mathcal{D}^n$

Definition: Let $\alpha \in \mathbb{C}^n$ denote a set of Langlands parameters. We define the power function $I_\alpha : U_n(\mathbb{R})\backslash\mathfrak{h}^n \to \mathbb{C}$ by

$$\boxed{I_\alpha(g) := \prod_{i=1}^{n-1}\prod_{j=1}^{n-1} y_i^{b_{i,j}\frac{1+\alpha_j-\alpha_{j+1}}{n}}}$$

$$g = xy \in \mathfrak{h}^n, \qquad b_{i,j} = \begin{cases} i \cdot j & \text{if } i+j \le n, \\ (n-i)(n-j) & \text{if } i+j \ge n. \end{cases}$$

The eigenfunction I_α

For Langlands parameters $\alpha \in \mathbb{C}^n$ we have

$$\boxed{\delta I_\alpha = \lambda_\delta \cdot I_\alpha \qquad \big(\forall\, \delta \in \mathcal{D}^n\big)}$$

and for the Laplacian Δ we have

$$\lambda_\Delta = \frac{n^3 - n}{24} - \frac{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2}{2}.$$

Character of the unipotent group

Let $u \in U_n$ be given by

$$u = \begin{pmatrix} 1 & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ & 1 & u_{2,3} & \cdots & u_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & u_{n-1,n} \\ & & & & 1 \end{pmatrix}$$

For $L = (\ell_1, \ell_2, \ldots, \ell_{n-1}) \in \mathbb{Z}^{m-1}$, we may define a character:

$$\boxed{\psi_L(u) := e^{2\pi i \left(\ell_1 u_{1,2} + \cdots + \ell_{n-1} u_{n-1,n}\right)}.}$$

Here

$$\psi_L(u \cdot u') = \psi_L(u)\psi_L(u'), \qquad (u, u' \in U_n).$$

Whittaker Functions

Given Langlands parameters $\alpha \in \mathbb{C}^n$ and a character ψ of $U_n(\mathbb{R})$ there is a unique Whittaker function

$$\boxed{W_\alpha : \mathfrak{h}^n \to \mathbf{C}}$$

Whittaker Function Properties

- $\delta W_\alpha = \lambda_\delta \cdot W_\alpha, \qquad (\forall\, \delta \in \mathcal{D}^n),$
- $W_\alpha(ug) = \psi(u) \cdot W_\alpha(g), \quad (\forall u \in U_n(\mathbb{R}),\, g \in GL(n, \mathbb{R})),$
- W_α is invariant under all permutations of $\alpha = \{\alpha_1, \ldots, \alpha_n\}$,
- W_α has holomorphic continuation to all $\alpha \in \mathbb{C}^n$,
- $W_\alpha(y)$ has rapid decay in $y_i \to \infty$ where $y = \operatorname{diag}(y_1, y_2, \cdots y_n)$,
- $W_\alpha(y)$ has prescribed polynomial asymptotics as all $y_i \to 0$.

Fourier-Whittaker expansion of cusp forms ϕ (Shalika-Piatetski-Shapiro)

Assume $\delta\,\phi = \lambda_\alpha \cdot \phi$ for all $\delta \in \mathcal{D}^n$.

Let $M = (m_1, \ldots, m_{n-1}) \in \mathbb{Z}^{n-1}$ and $\Gamma_{n-1} = \mathrm{SL}(n-1, \mathbb{Z})$.

$$\boxed{\phi(g) = \sum_{\gamma \in U_{n-1}\backslash\Gamma_{n-1}} \sum_{M \neq 0} \frac{A(M)}{\prod_{k=1}^{n-1} |m_k|^{\frac{k(n-k)}{2}}} W_\alpha\left(M^* \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right)}$$

where $g \in \mathfrak{h}^n$ and $M^* = \begin{pmatrix} m_1 \cdots m_{n-2}|m_{n-1}| & & & \\ & \ddots & & \\ & & m_1 & \\ & & & 1 \end{pmatrix}$.

$A(m_1, \ldots, m_{n-1})$ is called the M^{th} Fourier coefficient of ϕ.

L-function associated to a Hecke cusp form ϕ

L-function

$$L(s,\phi) = \sum_{m=1}^{\infty} \frac{A(m,1,\dots,1)}{m^s}$$

Euler Product

$$L(s,\phi) =$$

$$= \prod_p \left(1 - \frac{A(p,1,\dots,1)}{p^s} + \frac{A(1,p,1,\dots,1)}{p^{2s}} - \frac{A(1,1,p,\dots,1)}{p^{3s}} \right.$$

$$\left. + \quad \cdots \quad + (-1)^{n-1}\frac{A(1,,\dots,1,p)}{p^{(n-1)s}} + \frac{(-1)^n}{p^{ns}}\right)^{-1}$$

Functional Equation of $L(s, \phi)$

$L(s, \phi)$ is a degree n L-function. This means the completed L-function has n Gamma factors and satisfies the functional equation

$$\boxed{L^*(s, \phi) := \pi^{-\frac{ns}{2}} \prod_{i=1}^{n} \Gamma\left(\frac{s - \alpha_i}{2}\right) L(s, \phi) = L^*(1 - s, \widetilde{\phi})}$$

where $\widetilde{\phi}$ denotes the dual form which has M^{th} Fourier coefficient (for $M = (m_1, m_2, \ldots, m_{n-1})$) given by $A(m_{n-1}, m_{n-2}, \ldots, m_1)$.

Fourier coefficients of a cusp form

Let $M = (m_1, \ldots, m_{n-1})$ and $\mathbf{1} = (1, \ldots, 1)$. Then

$$A_\phi(M) = \underbrace{A_\phi(\mathbf{1})}_{\text{first coeff.}} \cdot \underbrace{\lambda_\phi(M)}_{M^{th}\text{ Hecke coeff.}}$$

where $\lambda_\phi(\mathbf{1}) = 1$.

First Coefficient of a cusp form ϕ

$$\left|A_\phi(\mathbf{1})\right|^2 = \frac{\langle \phi, \ \phi \rangle}{\mathrm{Vol}\left(\Gamma \backslash \mathfrak{h}^n\right) \cdot L(1, \phi_j, \mathrm{Ad}) \prod\limits_{1 \le j \ne k \le n} \Gamma\left(\frac{1+\alpha_j - \alpha_k}{2}\right)}$$

where $L(s, \phi_j, \mathrm{Ad}) := \frac{L\left(s, \, \phi_j \times \overline{\phi_j}\right)}{\zeta(s)}$.

Parabolic subgroups

Associated to a partition $n = n_1 + \cdots + n_r$, we have a standard parabolic subgroup

$$\mathcal{P} := \mathcal{P}_{n_1,n_2,\ldots,n_r} := \left\{ \begin{pmatrix} GL_{n_1} & * & \cdots & * \\ 0 & GL_{n_2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & GL_{n_r} \end{pmatrix} \right\} = N^{\mathcal{P}} \cdot M^{\mathcal{P}}$$

with nilpotent radical and Levi subgroup

$$N^{\mathcal{P}} := \left\{ \begin{pmatrix} I_{n_1} & * & \cdots & * \\ 0 & I_{n_2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n_r} \end{pmatrix} \right\}, \quad M^{\mathcal{P}} := \left\{ \begin{pmatrix} GL_{n_1} & 0 & \cdots & 0 \\ 0 & GL_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & GL_{n_r} \end{pmatrix} \right\}.$$

Fix a parabolic subgroup $\mathcal{P}$. Every $g \in \mathfrak{h}^n$ can be put in the form

$$\boxed{g = \mathfrak{n}(g) \cdot \mathfrak{m}(g)} \qquad \Big(\mathfrak{n}(g) \in N^{\mathcal{P}},\ m(g) \in M^{\mathcal{P}}\Big).$$

Here

$$\mathfrak{m}(g) = \begin{pmatrix} \mathfrak{m}_1(g) & 0 & \cdots & 0 \\ 0 & \mathfrak{m}_2(g) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathfrak{m}_r(g) \end{pmatrix}$$

where $\mathfrak{m}_i(g) \in GL_{n_i}$.

Langlands Eisenstein Series for $GL(n,\mathbb{R})$

Let $n > 2$, $\underbrace{\boxed{n = n_1 + n_2 + \cdots + n_r}}_{\text{partition}}$ and $\mathcal{P} := \underbrace{\mathcal{P}_{n_1,n_2,\ldots,n_r}}_{\text{parabolic subgroup}}$.

Let $s = (s_1, s_2, \ldots, s_r) \in \mathbb{C}$ where $\sum_{i=1}^{r} n_i s_i = 0$.

Power function on a parabolic subgroup

Define the power function $|\ |_{\mathcal{P}}^{s} : \mathfrak{h}^n \to \mathbb{C}$ by

$$\boxed{|g|_{\mathcal{P}}^{s} := \prod_{j=1}^{r} \left|\det(\mathfrak{m}_i(g))\right|^{s_j}} \qquad \Big(g = \mathfrak{n}(g)\,\mathfrak{m}(g)\,k \in GL(n,\mathbb{R})\Big).$$

Here $K = O(n,\mathbb{R})$. Note that $\sum_{i=1}^{r} n_i s_i = 0$ guarantees that $|\ |_{\mathcal{P}}^{s}$ is invariant under scalar multiplication.

$\boxed{\phi_i : \mathfrak{h}^{n_i} \to \mathbb{C}}$ are automorphic forms for $\mathrm{SL}(n_i, \mathbb{Z})$, $i = 1, 2, \ldots, r$.

(Automorphic form Φ associated to a parabolic $\mathcal{P}$)

Define an automorphic form $\Phi := (\phi_1, \ldots, \phi_r)$ on $\mathfrak{h}^n$ by the recipe

$$\boxed{\Phi(\mathfrak{n}\,\mathfrak{m}\,k) := \prod_{i=1}^{r} \phi_i(\mathfrak{m}_i)}$$

where $\mathfrak{n} \in N^{\mathcal{P}}$, $\mathfrak{m} \in M^{\mathcal{P}}$, $k \in K = O(n, \mathbb{R})$, and

$$\mathfrak{m} = \begin{pmatrix} \mathfrak{m}_1 & 0 & \cdots & 0 \\ 0 & \mathfrak{m}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathfrak{m}_r \end{pmatrix}, \qquad \Big(\mathfrak{m}_i \in GL(n_i, \mathbb{R})\Big),$$

DEFINITION: Langlands Eisenstein Series

Let $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ with $n > 2$. Consider a partition $n = n_1 + \cdots + n_r$ with associated parabolic subgroup $\mathcal{P}$. Let $s = (s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$ where $\sum_{i=1}^{r} n_i s_i = 0$.

The Langlands Eisenstein series determined by this data is defined by:

$$\boxed{E_{\mathcal{P},\Phi}(g, s) := \sum_{\gamma \in (\mathcal{P} \cap \Gamma)\backslash\Gamma} \Phi(\gamma g) \cdot |\gamma g|_{\mathcal{P}}^{s}}$$

Langlands Spectral Decomposition

Theorem (Langlands)

Let $\phi_1, \phi_2, \dots$ be an orthogonal basis of Maass forms for $\mathrm{SL}(n,\mathbb{Z})$*. Assume that $F \in \mathcal{L}^2(\mathrm{SL}(n,\mathbb{Z})\backslash\mathfrak{h}^n)$ are orthogonal to the residual spectrum. Then for $g \in \mathrm{GL}(n,\mathbb{R})$ we have*

$$F(g) = \sum_{j=1}^{\infty} \langle F, \phi_j \rangle \frac{\phi_j(g)}{\langle \phi_j, \phi_j \rangle} \; + \; \sum_{\mathcal{P}} \sum_{\Phi} c_{\mathcal{P}}$$

$$\cdot \int\limits_{\mathrm{Re}(s_1)=0} \cdots \int\limits_{\mathrm{Re}(s_{r-1})=0} \Big\langle F, E_{\mathcal{P},\Phi}(*\,, s) \Big\rangle E_{\mathcal{P},\Phi}(g\,, s)\; ds_1 \cdots ds_{r-1}$$

where the sum over $\mathcal{P}$ ranges over parabolics associated to partitions $n_1 + \cdots + n_r = n$, and the sum over Φ ranges over an orthonormal basis of Maass forms associated to $\mathcal{P}$. Here $s = (s_1, \dots, s_r)$ where $\sum\limits_{k=1}^{r} n_k s_k = 0$ for the partition $\sum\limits_{k=1}^{r} n_k = n$.

Minimal Parabolic Eisenstein Series

$\mathcal{P}_{\text{Min}}$ corresponds to the partition $n = 1 + 1 + \cdots + 1$. Let $s = (s_1, \ldots, s_n)$ with $\sum_{i=1}^{n} s_i = 0$.

$$\boxed{E_{\mathcal{P}_{\rm Min}}(g,s) := \sum_{\gamma \in (\mathcal{P}_{\rm Min} \cap \Gamma)\backslash\Gamma} |\gamma g|^{s}_{\mathcal{P}_{\rm Min}}} \qquad \Big(g \in GL(n,\mathbb{R}), \;\; \Re(s) \gg 1 \Big).$$

Non-constant Fourier coefficients of $E_{\mathcal{P}_{\rm Min}}$

Let $M = (m_1, \ldots, m_{n-1})$. Then

$$A_{\mathcal{P}_{\rm Min}}(M,s) = A_{\mathcal{P}_{\rm Min}}\big((1,\ldots,1),s\big) \cdot \lambda_{\mathcal{P}_{\rm Min}}(M,s),$$

$$A_{\mathcal{P}_{\rm Min}}(M,s) \;=\; \underbrace{A_{\mathcal{P}_{\rm Min}}(\mathbf{1},s)}_{\text{first coeff.}} \cdot \underbrace{\lambda_{\mathcal{P}_{\rm Min}}(M,s)}_{M^{th}\text{ Hecke coeff.}}$$

Fourier coefficients of Minimal Parabolic Eisenstein Series

Theorem (Selberg, Maass, Terras, Langlands, Shahidi)

Let $E_{\mathcal{P}_{\mathrm{Min}}}(g,s)$ *have Langlands parameters* $\alpha = (\alpha_1(s), \ldots, \alpha_n(s))$. *Then for* $m = 0, 1, 2, 3, \ldots,$

$$\boxed{\lambda_{\mathcal{P}_{\mathrm{Min}}}\big((m,1,\ldots,1),s\big) = \sum_{c_1 c_2 \cdots c_n = m} c_1^{\alpha_1(s)} c_2^{\alpha_2(s)} \cdots c_n^{\alpha_n(s)}}$$

$$\boxed{A_{\mathcal{P}_{\mathrm{Min}}}\big((1,\ldots,1),s\big) = \prod_{1 \le j < k \le n} \zeta^*\Big(1 + \alpha_j(s) - \alpha_k(s)\Big)^{-1}}$$

where $\zeta^*(w) = \pi^{-\frac{w}{2}}\Gamma\left(\frac{w}{2}\right)\zeta(w) = \zeta^*(1-w), \quad (w \in \mathbb{C}).$

The m^{th} Hecke eigenvalue of $E_{\mathcal{P},\Phi}(g,s)$

Theorem (G)

- Partition: $n = n_1 + \cdots n_r$.
- $s = (s_1, \ldots, s_r) \in \mathbb{C}^r$ with $n_1 s_1 + \cdots + n_r s_r = 0$.
- Hecke operator: T_m for $m = 1, 2, 3, \ldots$ Then

$$\boxed{T_m \, E_{\mathcal{P},\Phi}(g,s) \; = \; \lambda_{E_{P,\Phi}}\big((m,1,\ldots,1),s\big) \cdot E_{\mathcal{P},\Phi}(g,s)}$$

where

$$\lambda_{E_{P,\Phi}}\big((m,1,\ldots,1),s\big) = \sum_{\substack{1 \le c_1, c_2, \ldots, c_r \in \mathbb{Z} \\ c_1 c_2 \cdots c_r = m}} \lambda_{\phi_1}(c_1) \cdots \lambda_{\phi_r}(c_r)$$

$$\cdot \, c_1^{s_1 + N_1 + \frac{n_1 - n}{2}} \, c_2^{s_2 + N_2 + \frac{n_2 - n}{2}} \cdots c_r^{s_r + N_r + \frac{n_r - n}{2}},$$

and $N_1 = 0$, $N_i = n_1 + n_2 + \cdots n_{i-1}$ for $i \ge 1$.

Here $\lambda_{\phi_i}(c_i)$ is the eigenvalue of the Hecke operator T_{c_i} acting on ϕ_i.

(The first Fourier coefficient of $E_{\mathcal{P},\Phi}$)

Theorem (Stade-Woodbury-G)

Let $\mathcal{P} = \mathcal{P}_{n_1,\ldots,n_r}$, $s = (s_1,\ldots,s_r)$ *with* $n_1s_1+\cdots+n_rs_r = 0$. *Assume that each Maass form* ϕ_k *(with* $1 \le k \le r$*) occurring in* Φ *has Langlands parameters* $\alpha^{(k)} := (\alpha_{k,n_1},\ldots,\alpha_{k,n_k})$ *with the convention that if* $n_k = 1$ *then* $\alpha_{k,1} = 0$.

Then the first coefficient of $E_{\mathcal{P},\Phi}$ *is given by*

$$A_{E_{\mathcal{P},\Phi}}((1,\ldots,1),s) = \prod_{\substack{k=1\\ n_k\neq 1}}^{r} L^*\big(1,\mathrm{Ad}\,\phi_k\big)^{-\frac{1}{2}} \prod_{1\le j<\ell\le r} L^*\big(1+s_j-s_\ell,\ \phi_j\times\phi_\ell\big)^{-1}$$

up to a non-zero constant factor with abs. value depending on n.

Here

$$L^*(1,\ \mathrm{Ad}\ \phi_k) = L(1,\ \mathrm{Ad}\ \phi_k)\prod_{1\le i\neq j\le n_k}\Gamma\left(\frac{1+\alpha_{k,i}-\alpha_{k,j}}{2}\right)$$

and

$$L^*(1+s_j-s_\ell,\ \phi_j\times\phi_\ell) = \begin{cases} L^*(1+s_j-s_\ell,\ \phi_j) & \text{if } n_\ell = 1 \text{ and } n_j\neq 1,\\ L^*(1+s_j-s_\ell,\ \phi_\ell) & \text{if } n_j = 1 \text{ and } n_\ell\neq 1,\\ \zeta^*(1+s_j-s_\ell) & \text{if } n_j = n_\ell = 1.\end{cases}$$

Otherwise, $L^*(1+s_j-s_\ell,\phi_j\times\phi_\ell)$ is the completed Rankin-Selberg L-function.

EXAMPLE: Langlands Eisenstein series for $SL(4, \mathbb{Z})$

There are 4 standard non-associate parabolic subgroups associated to the partitions:

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

Let $E_{\mathcal{P},\Phi}(*, s)$ have Langlands parameters $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ which depend on s and the Langlands parameters of Φ.

EXAMPLE: The first Fourier coefficient of $E_{\mathcal{P}_{2,2},\Phi}$

Proposition (Shahidi-Woodbury-G)

The first coefficient of $E_{\mathcal{P}_{2,2},\Phi}$, where ϕ_1, ϕ_2 are Maass forms of norm 1 on GL(2) with spectral parameters $\frac{1}{2}+v$, $\frac{1}{2}+v'$, is given by

$$\left(L(1, \mathrm{Ad}\ \phi_1)^{\frac{1}{2}}\, L(1, \mathrm{Ad}\ \phi_2)^{\frac{1}{2}}\, \Gamma\left(\tfrac{1}{2}+v\right)\ \Gamma\left(\tfrac{1}{2}+v'\right) L^*\left(1+2s_1, \phi_1 \times \phi_2\right) \right)^{-1}$$

up to a constant non-zero factor.

EXAMPLE: The first Fourier coefficient of $E_{\mathcal{P}_{2,1,1},\Phi}$

$$s = \left(1+s_1, -\frac{1}{2}+s_2, s_3\right), \quad \alpha = (s_1+v, s_1-v, s_2, -2s_1-s_2)$$

Proposition

The first coefficient of $E_{\mathcal{P}_{2,1,1},\Phi}(g,s)$, where ϕ is a Maass form of norm 1 on GL(2) with spectral parameter $\frac{1}{2}+v$, is given by

$$\Big(L(1, \mathrm{Ad}\ \phi)^{\frac{1}{2}}\, \Gamma\left(1/2+v\right) \zeta^*(1+2s_1+2s_2) L^*(1+s_1-s_2,\phi) \cdot L^*(1+3s_1+s_2,\phi)\Big)^{-1}$$

up to a constant non-zero factor.

EXAMPLE: The first Fourier coefficient of $E_{\mathcal{P}_{3,1},\Phi}$

$$s = \left(1/2 + s_1,\ -3/2 - 3s_1\right)$$

$$\alpha = (s_1 + 2v + v', s_1 - v + v', s_1 - v - 2v', -3s_1)$$

Proposition

The first coefficient of $E_{\mathcal{P}_{3,1},\Phi}(g,s)$, where ϕ is a Maass form of norm 1 on GL(3) with Langlands parameter $(2v + v', -v + v', -v - 2v')$, is given by

$$\left(L(1, \mathrm{Ad}\ \phi)^{\frac{1}{2}}\, \Gamma\left(\frac{1+3v}{2}\right) \Gamma\left(\frac{1+3v'}{2}\right) \Gamma\left(\frac{1+3v+3v'}{2}\right) \cdot L^*(1+4s_1, \phi) \right)^{-1}$$

up to a constant non-zero factor.