Invitation to pair correlation problems

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Diophantine approximation

Theorem (Dirichlet approximation theorem)

Let α be irrational. There are infinitely many coprime solutions (p, q) to the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Kronecker approximation theorem

Theorem (Kronecker approximation theorem, 1884)

Let α be irrational. Then the sequence

$$(n\alpha)_{n \geq 1} \mod 1$$

is dense in $[0, 1]$.

Equidistribution I

Theorem (Bohl, Sierpiński, Weyl, ca. 1909)

Let α be irrational. Then the sequence

$$(n\alpha)_{n \geq 1} \quad \text{mod } 1$$

is uniformly distributed mod 1 (equidistributed), i.e.

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_I(\{n\alpha\}) \to \text{meas}(I)$$

for all intervals $I \subset [0,1]$.

Fact: an i.i.d. random sequence is equidistributed, almost surely.

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Equidistribution II

The following sequences are equidistributed:

- $(n^d \alpha)_{n \geq 1}, \qquad \alpha \notin \mathbb{Q},\ d \geq 1$
- $(n^\alpha)_{n \geq 1}, \qquad \alpha > 0,\ \alpha \notin \mathbb{N}$
- $((\log n)^\alpha)_{n \geq 1}, \qquad \alpha > 1.$

Equidistribution III

Theorem (Weyl criterion, 1916)

A sequence $(x_n)_{n\geq 1}$ is u.d. mod 1 if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0$$

for all $h \in \mathbb{Z},\ h \neq 0$.

- Weyl, H. Über die Gleichverteilung von Zahlen mod. Eins. Math. Ann. 77 (1916), no. 3, 313–352.

Equidistribution IV

Claim: $(n\alpha)_{n\geq 1}$ is u.d. mod 1 if $\alpha \notin \mathbb{Q}$.

Proof: Let $h \neq 0$. Then

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h n \alpha} = \frac{1}{N} \cdot \frac{e^{2\pi i h \alpha} - e^{2\pi i h (N+1)\alpha}}{1 - e^{2\pi i h \alpha}}.$$

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$\to 0$

bounded

Equidistribution V

A number α is called a *normal number* (in base $b \geq 2$) if

$$(b^n \alpha)_{n \geq 1} \qquad \text{is u.d. mod } 1.$$

Theorem (Borel, 1909)

Almost all numbers are normal.

It is usually very difficult to find out if a number is normal or not.

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It is usually very difficult to find out if a number is normal or not.

Equidistribution VI

Theorem (Weyl, 1916)

Let $(a_n)_{n \geq 1}$ be a sequence of distinct positive integers. Then the sequence

$$(a_n \alpha)_{n \geq 1}$$

is u.d. mod 1 for almost all α.

weitere Zahlen in ihr enthalten sind. Wenn ich nun freilich glaube, daß man den Wert solcher Sätze, in denen eine unbestimmte Ausnahmemenge vom Maße 0 auftritt, nicht eben hoch einschätzen darf, möchte ich diese Behauptung hier doch kurz begründen. Mein Beweis beruht auf dem

Figure: Weyl, H. Über die Gleichverteilung von Zahlen mod. Eins. Math. Ann. 77 (1916), no. 3, 313–352.

Pair correlation for the zeta function I

Conjecture (Montgomery's pair correlation conjecture, 1973)

Assume RH. Then

$$\lim_{T\to\infty} \frac{2\pi}{T\log T} \sum_{0\le\gamma\neq\gamma'\le T} \mathbb{1}_{\left[-\frac{2\pi s}{\log T},\frac{2\pi s}{\log T}\right]}(\gamma-\gamma') = \int_{-s}^{s} 1-\left(\frac{\sin \pi u}{\pi u}\right)^2 du$$

for all $s \geq 0$.

In a certain standard terminology the Conjecture may be formulated as the assertion that $1-((\sin \pi u)/\pi u)^2$ is the pair correlation function of the zeros of the zeta function. F. J. Dyson has drawn my attention to the fact that the eigenvalues of a random complex Hermitian or unitary matrix of large order have precisely the same pair correlation function (see [**3**, equations (6.13), (9.61)]). This means that the Conjecture fits well with the view that there is a linear operator (not yet discovered) whose eigenvalues characterize the zeros of the zeta function. The eigenvalues of a random real symmetric matrix of large order have a different pair

From: H.L. Montgomery, The pair correlation of zeros of the zeta function. Analytic number theory (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972), pp. 181–193. Amer. Math. Soc. 1973.

Pair correlation for the zeta function II

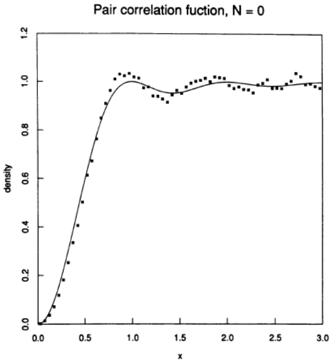

FIGURE 1

Pair correlation of zeros of the zeta function. Solid line: GUE prediction. Scatter plot: empirical data based on zeros γ_n, $1 \leqslant n \leqslant 10^5$.

From: A.M. Odlyzko, On the distribution of spacings between zeros of the zeta function. Math. Comp. 48 (1987), no. 177, 273–308.

Pair correlation for the zeta function III

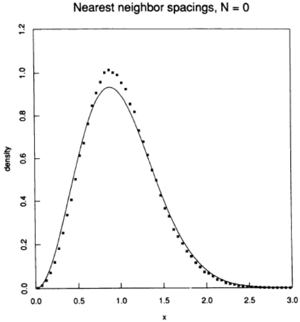

FIGURE 3

Probability density of the normalized spacings δ_n. *Solid line: GUE prediction. Scatter plot: empirical data based on zeros* γ_n, $1 \leqslant n \leqslant 10^5$.

From: A.M. Odlyzko, On the distribution of spacings between zeros of the zeta function. Math. Comp. 48 (1987), no. 177, 273–308.

Further reading:

• J.B. Conrey, A. Ghosh, S.M. Gonek, A note on gaps between zeros of the zeta function. Bull. London Math. Soc. 16 (1984), no. 4, 421–424.
• Z. Rudnick, P. Sarnak, Zeros of principal L-functions and random matrix theory. A celebration of John F. Nash, Jr. Duke Math. J. 81 (1996), no. 2, 269–322.
• N.M. Katz, P. Sarnak, Zeroes of zeta functions and symmetry. Bull. Amer. Math. Soc. (N.S.) 36 (1999), no. 1, 1–26.
• J.P. Keating, N.C. Snaith, Random matrix theory and $\zeta(1/2 + it)$. Comm. Math. Phys. 214 (2000), no. 1, 57–89.

• P. Bourgade, Tea time in Princeton. Harvard College Math. Review, 2012.
`https://math.nyu.edu/~bourgade/papers/TeaTime.pdf`.

The Berry–Tabor conjecture I

Proc. R. Soc. Lond. A. **356**, 375–394 (1977)
Printed in Great Britain

Level clustering in the regular spectrum

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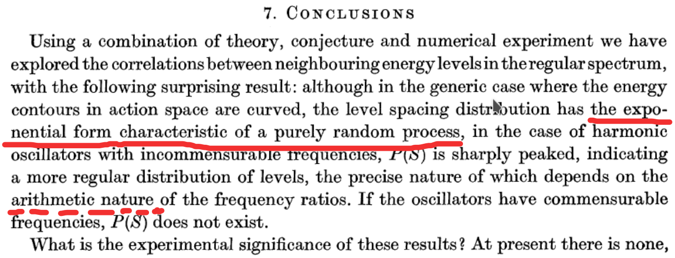
and M. Tabor‡
*H. H. Wills Physics Laboratory, Tyndall Avenue, Bristol BS*8 1*TL*

(*Communicated by J. M. Ziman, F.R.S. — Received* 1 *November* 1976)

From: M.V. Berry and M. Tabor, Level clustering in the regular spectrum. Proc. R. Soc. Lond. A 356 (1977), 375–394

The Berry–Tabor conjecture II


7. CONCLUSIONS

Using a combination of theory, conjecture and numerical experiment we have explored the correlations between neighbouring energy levels in the regular spectrum, with the following surprising result: although in the generic case where the energy contours in action space are curved, the level spacing distribution has the exponential form characteristic of a purely random process, in the case of harmonic oscillators with incommensurable frequencies, $P(S)$ is sharply peaked, indicating a more regular distribution of levels, the precise nature of which depends on the arithmetic nature of the frequency ratios. If the oscillators have commensurable frequencies, $P(S)$ does not exist.

What is the experimental significance of these results? At present there is none,

From: M.V. Berry and M. Tabor, Level clustering in the regular spectrum. Proc. R. Soc. Lond. A 356 (1977), 375–394

The Berry–Tabor conjecture III

4. THE GENERIC CASE: NUMERICAL EXPERIMENTS

Given any 'Hamiltonian' $U(\boldsymbol{m})$, it is an easy matter to compute all levels $\boldsymbol{m}$ for which $U(\boldsymbol{m})$ is less than some large number $U_{\max}$, arrange them in order of increasing U, compute the differences S between the U-values of adjacent levels and plot the spacing distribution $P(S)$ as a histogram. In our experiments we first took $U_{\max} = 5000$ and then repeated the calculations with $U_{\max} = 10\,000$, to check that the distributions $P(S)$ were stable (they were, in every case except one which will be mentioned in § 6).

The first experiments were with two-dimensional boxes (case II of § 2) with sides a_1 and a_2; setting $a_1^2/a_2^2 \equiv a/b$ equation (2.11) gives

$$U^{\mathrm{II}}(\boldsymbol{m}) = \tfrac{1}{4}\pi(m_1^2\sqrt{(b/a)} + m_2^2\sqrt{(a/b)}). \qquad (4.1)$$

Figure 2a shows $P(S)$ for $a/b = \sqrt{2}$. The exponential distribution (3.17) is clearly a good fit. Calculations for $a/b = \sqrt{3}, \sqrt{5}, \sqrt{7}$ and $\frac{1}{2}(\sqrt{5}-1)$ show essentially the same

From: M.V. Berry and M. Tabor, Level clustering in the regular spectrum. Proc. R. Soc. Lond. A 356 (1977), 375–394

The Berry–Tabor conjecture IV

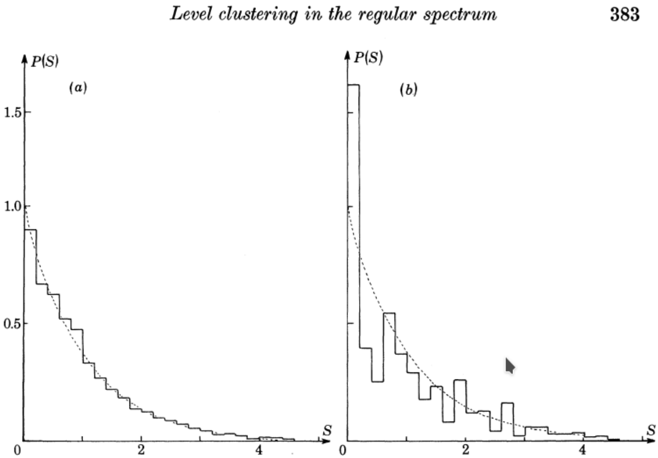


FIGURE 2. Distributions of spacings of lowest 5000 levels for particles in two-dimensional boxes with side ratio a^2/b^2 where (a) $a/b = \sqrt{2}$ (b) $a/b = \frac{5}{7}$. The dotted lines show the exponential distribution.

From: M.V. Berry and M. Tabor, Level clustering in the regular spectrum. Proc. R. Soc. Lond. A 356 (1977), 375–394

The Berry–Tabor conjecture V

Now the best sequence of rational approximations to α is obtained in terms of the successive convergents of its *simple continued fraction*. This is

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} \equiv [a_1 a_2 a_3 \ldots], \tag{5.8}$$

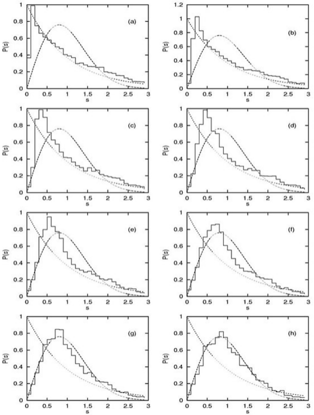
where the quantities a_i are positive integers, uniquely defined by the integral parts of the successive reciprocals of the non-integral parts of α. The successive convergents are the rational fractions

$$p_k/q_k = [a_1 a_2 a_3 \ldots a_k]. \tag{5.9}$$

They are best approximations to α in the sense that no rational fraction p/q with $q \leqslant q_k$ lies closer to α than p_k/q_k. This is proved in all texts on arithmetic, e.g. the classic by Khinchin (1964) or the more modern presentation by Drobot (1964). A physical interpretation of this result is given by Klein (1932): if pegs are attached

From: M.V. Berry and M. Tabor, Level clustering in the regular spectrum. Proc. R. Soc. Lond. A 356 (1977), 375–394

The Berry–Tabor conjecture V



Nearest neighbor distribution for Rydberg atom energy level spectra in an electric field as quantum defect is increased from 0.04 (a) to 0.32 (h). The system becomes more chaotic as dynamical symmetries are broken by increasing the quantum defect; consequently, the distribution evolves from nearly a Poisson distribution (a) to that of Wigner's surmise (h).

From: Wikipedia `https://en.wikipedia.org/wiki/Quantum_chaos`

The Berry–Tabor conjecture VI

The upshot (for the number theorist):

For the local distribution of many sequences of arithmetic origin, we should expect to see *Poissonian* behavior.

- J. Marklof, The Berry–Tabor conjecture. European Congress of Mathematics, Vol. II (Barcelona, 2000), 421–427.

Pair correlation: definitions

Definition

Let $(x_n)_{n\geq 1}$ be a sequence in the unit interval. We say that the pair correlation of the sequence is Poissonian if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1\leq m\neq n\leq N} \mathbb{1}_{[-\frac{s}{N},\frac{s}{N}]}(x_m - x_n) = 2s$$

for all $s \geq 0$.

Note:

$$2s = \int_{-s}^{s} 1 \, du.$$

Fact: an i.i.d. random sequence has PPC, almost surely.

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Pair correlation: definitions II

Definition

Let $(x_n)_{n\geq 1}$ be a sequence in the unit interval. We say that the distribution of neighbor gaps is Poissonian if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1\leq n\leq N-1} \mathbb{1}_{[0,\frac{s}{N}]} \left(x_{(n+1,N)} - x_{(n,N)} \right) = \int_0^s e^{-u}\, du.$$

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Fact: an i.i.d. random sequence has Poissonian neighbor spacings, almost surely.

Pair correlation: definitions III

Fact: a sequence $(x_n)_{n \geq 1}$ in $[0, 1]$ which has PPC is necessarily equidistributed. (The converse is generally false.)

• C. Aistleitner, T. Lachmann, F. Pausinger, Pair correlations and equidistribution. J. Number Theory 182 (2018), 206–220.
• S. Grepstad, G. Larcher, On pair correlation and discrepancy. Arch. Math. (Basel) 109 (2017), no. 2, 143–149.
• J. Marklof, Pair correlation and equidistribution on manifolds. Monatsh. Math. 191 (2020), no. 2, 279–294.
• S. Steinerberger, Localized quantitative criteria for equidistribution. Acta Arith. 180 (2017), no. 2, 183–199.

Pair correlation – the metric theory I

Theorem (Rudnick–Sarnak, 1998)

Let $d \geq 2$. The sequence

$$(n^d \alpha)_{n \geq 1} \qquad \text{mod } 1$$

has PPC for almost all α.

Compare: the sequence $(n\alpha)_{n\geq 1}$ is equidistributed for all irrational α, but has PPC for no α.

• Z. Rudnick, P. Sarnak, The pair correlation function of fractional parts of polynomials. Comm. Math. Phys. 194 (1998), no. 1, 61–70.

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Pair correlation – the metric theory II

MR1628282 (99g:11088) Reviewed
Rudnick, Zeév (IL-TLAV); Sarnak, Peter (1-PRIN)
The pair correlation function of fractional parts of polynomials. (English summary)
Comm. Math. Phys. 194 (1998), no. 1, 61–70.
11J54 (11J71)
Review PDF | Clipboard | Journal | Article | Make Link

Citations
From References: 71
From Reviews: 10

Fix an integer $d \geq 2$. The authors study the problem of the spacings between members of the sequence αn^d modulo one, $n = 1, 2, \cdots$, for a given irrational number α. This is a finer question than that of the uniform distribution of such numbers which was demonstrated by H. Weyl.

The authors call the number α "Diophantine" if for every $\varepsilon > 0$ there is a positive constant $c = c(\alpha, \varepsilon)$ such that $|\alpha - p/q| > c/q^{2+\varepsilon}$ for all integers p and q. Thus we know by Roth's theorem that algebraic irrationals are Diophantine. The authors define in a natural way the normalized pair correlation for the numbers in question. They conjecture that for each Diophantine α the pair correlation function exhibits the same asymptotic behaviour as that given by a sequence of random points in the unit interval. Using Fourier methods they prove that, for each d, the conjecture is true for most α (in the sense of Lebesgue measure). They remark that they do not know of any proof which exhibits this expected behaviour for any specific α, although they do show that this behaviour does not hold for any extremely well approximable irrationals (and examples of these can be exhibited as was done by Liouville). Finally the authors give a brief discussion of the higher correlations.

The reviewer thinks that these are interesting and natural questions. It seems surprising that they have not been studied earlier.

Reviewed by John B. Friedlander

References

Pair correlation – the metric theory III

For a sequence of integers $(a_n)_{n \geq 1}$, let

$$E_N = \# \{(n_1, n_2, n_3, n_4) : \ a_{n_1} + a_{n_2} = a_{n_3} + a_{n_4}\} .$$

Theorem (A–Larcher–Lewko, Bourgain, 2017)

Let $(a_n)_{n \geq 1}$ be a sequence of positive integers. If $E_N \ll N^{3-\varepsilon}$, then

$$(a_n \alpha)_{n \geq 1} \qquad \mod 1$$

has PPC for almost all α. If $E_N \gg N^3$, then the sequence has PPC for almost no α.

- C. Aistleitner, G. Larcher, M. Lewko, Additive energy and the Hausdorff dimension of the exceptional set in metric pair correlation problems. With an appendix by Jean Bourgain. Israel J. Math. 222 (2017), no. 1, 463–485.

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Pair correlation – individual sequences I

Theorem (El-Baz–Marklof–Vinogradov, 2015)

The sequence $(\sqrt{n})_{n \in \mathbb{N} \setminus \square}$ *has PPC.*

The distribution of neighbor gaps is non-Poissonian (Elkies–McMullen, 2004).

Keywords: ergodic theory, Ratner's theorem, horocycle flows.

Radziwiłł–Technau announced a purely arithmetic proof.

- N.D. Elkies, C. McMullen, Gaps in $\sqrt{n}$ mod 1 and ergodic theory. Duke Math. J. 123 (2004), no. 1, 95–139.
- D. El-Baz, J. Marklof, I. Vinogradov, The two-point correlation function of the fractional parts of $\sqrt{n}$ is Poisson. Proc. Amer. Math. Soc. 143 (2015), no. 7, 2815–2828.

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Pair correlation – individual sequences II

Theorem (Lutsko–Sourmelidis–Technau, 2022)

Let $\beta < 14/41$. The sequence $(n^\beta)_{n\geq 1}$ has PPC.

Keywords: van der Corput's B-process.

- C. Lutsko, A. Sourmelidis, N. Technau, Pair Correlation of the Fractional Parts of αn^θ, arXiv:2106.09800, 2021.

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Open problems I

Open problem (Rudnick–Sarnak conjecture)

If α cannot be approximated too well by rationals, then the pair correlation of

$$(n^2\alpha)_{n\geq 1} \qquad \text{mod } 1$$

is Poissonian.

• D.R. Heath-Brown, Pair correlation for fractional parts of αn^2. Math. Proc. Cambridge Philos. Soc. 148 (2010), no. 3, 385–407.

• J.L. Truelsen, Divisor problems and the pair correlation for the fractional parts of $n^2\alpha$. Int. Math. Res. Not. IMRN 2010, no. 16, 3144–3183.

• J. Marklof, N. Yesha, Pair correlation for quadratic polynomials mod 1. Compos. Math. 154 (2018), no. 5, 960–983.

Open problems II

Open problem (Metric theory)

Give a sharp criterion on integer sequences $(a_n)_{n \geq 1}$ such that $(a_n \alpha)_{n \geq 1}$ has PPC for almost all α.

- C. Aistleitner, T. Lachmann, N. Technau, There is no Khintchine threshold for metric pair correlations. Mathematika 65 (2019), no. 4, 929–949.
- T.F. Bloom, A. Walker, GCD sums and sum-product estimates. Israel J. Math. 235 (2020), no. 1, 1–11.

Open problems III

Open problem (Metric theory for real sequences)

Give a sharp criterion on real sequences $(a_n)_{n\geq 1}$ such that $(a_n\alpha)_{n\geq 1}$ has PPC for almost all α.

• Z. Rudnick, N. Technau, The metric theory of the pair correlation function of real-valued lacunary sequences. Illinois J. Math. 64 (2020), no. 4, 583–594.
• C. Aistleitner, D. El-Baz, M. Munsch, A pair correlation problem, and counting lattice points with the zeta function. Geom. Funct. Anal. 31 (2021), no. 3, 483–512.

Open problems IV

Open problem (Correlations of higher order)

Show that $(n^2\alpha)_{n\geq 1}$ has Poissonian triple correlation for almost all α.

- N. Technau, A. Walker, The triple correlations of fractional parts of αn^2, arXiv:2005.01490, 2020.
- C. Lutsko, Long-range correlations of sequences modulo 1. J. Number Theory 234 (2022), 333–348.

Open problems V

Open problem (Small gaps)

Determine the size of the smallest gap of $(x_n)_{n \geq 1}$ for "interesting" sequences $(x_n)_{n \geq 1}$ (metric / non-metric).

- V. Blomer, J. Bourgain, M. Radziwiłł, Z. Rudnick, Small gaps in the spectrum of the rectangular billiard. Ann. Sci. Ec. Norm. Super. (4) 50(5) (2017), 1283–1300.
- Z. Rudnick, A metric theory of minimal gaps. Mathematika 64 (2018), no. 3, 628–636.
- C. Aistleitner, D. El-Baz, M. Munsch, Difference sets and the metric theory of small gaps. arXiv:2108.02227, 2021

Think: Diophantine approximation with restricted denominators.
Try to make $\|(a_m - a_n)\alpha\|$ small.

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Open problems VI

Open problem (Large gaps)

Determine the size of the largest *gap of* $(x_n)_{n \geq 1}$ *for "interesting" sequences* $(x_n)_{n \geq 1}$ *(metric / non-metric).*

• C. Aistleitner, V. Blomer, M. Radziwiłł, Triple correlation and long gaps in the spectrum of flat tori. arXiv:1809.07881, 2018

Open problems VII

Open problem (Higher dimension)

Develop a corresponding theory for multi-dimensional sequences.

• A. Hinrichs, L. Kaltenböck, G. Larcher, W. Stockinger, M. Ullrich, On a multi-dimensional Poissonian pair correlation concept and uniform distribution. Monatsh. Math. 190 (2019), no. 2, 333–352.
• S. Steinerberger, Poissonian pair correlation in higher dimensions. J. Number Theory 208 (2020), 47–58.
• T. Bera, M. Kumar Das, A. Mukhopadhyay, On higher dimensional Poissonian pair correlation. arXiv:2207.02584, 2022.

Open problems VIII

Open problem (Realizability questions)

For which functions $g(s)$ can there exist a sequence $(x_n)_{n\geq 1}$ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1\leq m\neq n\leq N} \mathbb{1}_{\left[-\frac{s}{N},\frac{s}{N}\right]}(x_m - x_n) = \int_{-s}^{s} g(u)\, du$$

for all $s \geq 0$.

- T. Kuna, J.L. Lebowitz, E.R. Speer, Realizability of point processes. J. Stat. Phys. 129 (2007), no. 3, 417–439.

Open problems IX

Open problem (Realizability questions)

Does Poissonian triple correlation imply Poissonian pair correlation?

Open problems X

Open problem (Realizability questions)

Is is possible that a sequence has Poissonian triple correlation, and the maximal gap among its first N elements is of size at most $\frac{C}{N}$.

• C. Aistleitner, V. Blomer, M. Radziwiłł, Triple correlation and long gaps in the spectrum of flat tori. arXiv:1809.07881, 2018

Fact: the minimal size of the largest gap of a sequence having PPC is at least $\frac{1.5}{N}$, but does not need to exceed $\frac{2}{N}$.

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