SOME ALGEBRAIC METHODS FOR ANALYSING MATRIX CONTINUED FRACTIONS

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ABSTRACT: Scalar continued fractions (CFs) have been playing an important role not only in number theory but also in other branches of mathematics. As a generalisation of scalar CFs, we try to consider matrix continued fractions (MCFs):

$$\cfrac{B_1}{A_1 + \cfrac{B_2}{A_2 + \cfrac{B_3}{A_3 + \ddots}}}, \tag{1}$$

where $A_k, B_k \in \mathcal{M}_n(\mathbb{C})$ and $\frac{B}{A} = A^{-1}B$.

However, this generalisation creates many problems. For instance, even if A_1, A_2, B_1, B_2 are all invertible, we cannot guarantee the invertibility of $A_1 + \frac{B_2}{A_2}$, so the 'partial convergents' of the MCF (1), which is obtained by stopping (1) in finite lengths, may not be defined properly. Another problem is that the knowledge on MCFs is limited compared to scalar CFs, so, for example, if one prepares a MCF (1), it may be difficult to decide if it is convergent or not (cf. [2]).

For dealing with these problems, in this presentation we will try to 'reduce' the MCF (1) into a simpler form so that we can say something more about (1), e.g. the convergence. Especially, if we are able to 'reduce' the MCF into a collection of scalar CFs, we can analyse the MCF much further because we have more knowledge on scalar CFs.

For the reduction of the MCF (1), in this presentation we will apply several algebraic tools: (generalised) Shemesh criterion, the ALS-criterion, the discriminant of an associative algebra and some construction methods such as by Eberly, Friedl and Rónyai (cf. [1]). The first three methods can be used to check the 'reducibility' of the MCF (1), and the latter ones can be used to explicitly compute the reduced form if the MCF is recognised to be completely reducible. An important point of using these tools is that they are 'effective' (or rational), i.e. they are step-by-step procedures (algorithms) and one can obtain the result in a finite number of steps.

The ALS-criterion and the discriminant of an associative algebra can be used to test the simultaneous block-diagonalisability of the matrices A_k, B_k in (1), so if all A_k, B_k are simultaneously block-diagonalised, the MCF (1) can be considered as a collection of MCFs with smaller dimensions, which

simplifies the analysis of the original MCF (1). Similarly, the generalised Shemesh criterion can be used to test the (complete) reducibility of the MCF so that the matrices A_k, B_k are simultaneously block-diagonalised into $(n-1)\times(n-1)$ and 1×1 blocks. If so, the MCF (1) on the one-dimensional subspace can be considered as a scalar CF. This is beneficial because the MCF (1) on the one-dimensional subspace can be considered as a scalar CF so we can analyse it well by applying known results for scalar CFs.

In this presentation, we will study the effective procedures and how we are able to use in order for the analysis of MCFs.

[1] T. Kamizawa, Open Syst. Infor. Dyn. **26(2)**, 2019, 1950010.

[2] T. Kamizawa, Asian-Eur. J. Math. 2021, DOI: 10.1142/S1793557122501418.